EECS 16A Designing Information Devices and Systems I
Fall 2022 Homework 1

1. Reading Assignment

For this homework, please read Note 0, Note 1A, and Note 1B. These will provide an overview of linear equations, augmented matrices, and Gaussian Elimination. You are always welcome and encouraged to read ahead as well. How does the content you read in these notes relate to what you've learned before? What content is unfamiliar or new?

Solution: Give yourself credit for any reasonable answer.

2. Counting Solutions

Learning Goal: *(This problem is meant to illustrate the different types of systems of equations. Some have a unique solution and others have no solutions or infinitely many solutions. We will learn in this class how to systematically figure out which of the three above cases holds.)*

For each of the following systems of linear equations, determine if there is a unique solution, no solution, or an infinite number of solutions. If there is a unique solution, find it. If there is an infinite number of solutions state this and give one solution. If there is no solution, explain why. **Use augmented matrices and show your work**.

(a)

$$\begin{aligned} 2x &+ 3y &= 5 \\ x &+ y &= 2 \end{aligned}$$

Solution:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 1 & 1 & 2 \end{array}\right] \text{using } \tfrac{1}{2}R_1 \to R_1$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \text{using } R_2 - R_1 \to R_2$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 1 \end{array}\right] \text{using } -2R_2 \to R_2$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \text{using } R_1 - \tfrac{3}{2}R_2 \to R_1$$

Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b)

$$\begin{aligned} x &+ y &+ z &= 3 \\ 2x &+ 2y &+ 2z &= 5 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right] \text{using } R_2 - 2R_1 \rightarrow R_2$$

No solution. The fact that there are fewer equations than there are unknowns immediately means that it is not possible to have a unique solution; however, this does not guarantee that there is a solution to begin with. From Gaussian Elimination, we can see that these equations are inconsistent since $0 \neq -1$. In other words, no values of x, y, and z can satisfy both equations simultaneously.

(c)

$$\begin{array}{rcrcrl} & - & y & + & 2z & = 1 \\ 2x & & & + & z & = 2 \end{array}$$

Solution:

Because there are two equations and three unknowns, we immediately see that there can be no unique solution. The question then becomes if there are an infinite number of solutions, or no solution at all.

$$\begin{aligned} \left[\begin{array}{ccc|c} 0 & -1 & 2 & 1 \\ 2 & 0 & 1 & 2 \end{array}\right] &\rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & -1 & 2 & 1 \end{array}\right] \text{swapping } R_1 \text{ and } R_2 \\ &\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & -1 & 2 & 1 \end{array}\right] \text{using } \tfrac{1}{2}R_1 \rightarrow R_1 \\ &\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & -2 & -1 \end{array}\right] \text{using } -R_2 \rightarrow R_2 \end{aligned}$$

We have arrived at reduced row echelon form. In this way we can explicitly see that z is a free variable (x and y depend on z and there are no constraints on the value of z). Thus there are an infinite number of solutions. The set of infinite solutions has the form (for some $z \in \mathbb{R}$):

$$x = 1 - \frac{1}{2}z$$
$$y = 2z - 1$$

To get full credit it is enough to state "Infinite solutions" *and give one possible solution* that fits the form above.

(d)

$$\begin{array}{rcrl} x & + & 2y & = 3 \\ 2x & - & y & = 1 \\ 3x & + & y & = 4 \end{array}$$

Solution:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 3 & 1 & 4 \end{array}\right] \text{using } R_2 - 2R_1 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -5 \end{array}\right] \text{using } R_3 - 3R1 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -5 \end{array}\right] \text{using } -\tfrac{1}{5}R_2 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \text{using } R_3 + 5R_2 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \text{using } R_1 - 2R_2 \rightarrow R_1$$

Unique solution, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$The system of linear equations at the end of the Gaussian Elimination above simply reads out

$$x = 1$$
$$y = 1$$
$$0 = 0$$

(e)

$$\begin{array}{rcrl} x & + & 2y & = 3 \\ 2x & - & y & = 1 \\ x & - & 3y & = -5 \end{array}$$

Solution:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 1 & -3 & -5 \end{array}\right] \text{using } R_2 - 2R_1 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -8 \end{array}\right] \text{using } R_3 - R_1 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -8 \end{array}\right] \text{using } -\tfrac{1}{5}R_2 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{array}\right] \text{using } R_3 + 5R_2 \rightarrow R_3$$

No solution. We can think of this to mean that there are no values of x and y which satisfy the conditions in all three equations simultaneously, because in order to satisfy all three equations, the last row $0 = -3$ would need to be true. Even though we have more equations than unknowns, that does not guarantee that a unique solution, or any solutions, exist.

3. Magic Square

In an $n \times n$ "magic square," all of the sums across each of the n rows, n columns, and 2 diagonals equal magic constant k. For example, in the below magic square, each row, column, and diagonal sums to 34.

4	14	15	1
9	7	6	12
5	11	10	8
16	2	3	13

The magic square is a classic math puzzle, and some of you may have solved these as children by guessing. However, it turns out they can be solved systematically by setting up a system of linear equations!

(a) How many linear equations can you write for an $n \times n$ magic square?

Solution:

$2n+2$, since there is one equation for each of the n rows, n columns, and 2 diagonals.

(b) For the generalized magic square below, write out a system of linear equations.

Hint: Set the sum of entries in each row, column, and diagonal equal to k.

x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	x_{23}
x_{31}	x_{32}	x_{33}

Solution:

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} &= k \\
x_{21} + x_{22} + x_{23} &= k \\
x_{31} + x_{32} + x_{33} &= k \\
x_{11} + x_{21} + x_{31} &= k \\
x_{12} + x_{22} + x_{32} &= k \\
x_{13} + x_{23} + x_{33} &= k \\
x_{11} + x_{22} + x_{33} &= k \\
x_{31} + x_{22} + x_{13} &= k
\end{aligned}
$$

(c) Now consider the following square, with some entries filled in. Substitute the known entries into the linear equations you wrote in part (b) to solve for the missing entries x_{11}, x_{12}, x_{32}. Please show the equations you use to solve; credit will not be given for solving by inspection.

2	x_{12}	6
9	5	1
x_{31}	x_{32}	8

Solution:

$$x_{31} + x_{32} + 8 = k \quad (1)$$
$$9 + 5 + 1 = k \quad (2)$$
$$2 + x_{12} + 6 = k \quad (3)$$
$$x_{31} + 9 + 2 = k \quad (4)$$
$$x_{32} + 5 + x_{12} = k \quad (5)$$
$$8 + 1 + 6 = k \quad (6)$$
$$x_{31} + 5 + 6 = k \quad (7)$$
$$2 + 5 + 8 = k \quad (8)$$

From Eq. 2, $k = 15$.
Substituting $k = 15$ back into Eq. 3, $x_{12} = 7$.
Similarly, substituting $k = 15$ back into Eq. 8, $x_{31} = 4$.
Finally, substituting $k = 15, x_{12} = 7$ into Eq. 5, we find $x_{32} = 3$.

(d) Suppose you now have a 'tomographic' magic square. This square is special in that the *product* of the *exponentials* of the elements sum to a constant. So, the equation for the first row might look like:

$$e^{x_{11}} \times e^{x_{12}} \times e^{x_{13}} = k$$

where k is the constant value of the magic square. Can you write out a system of linear equations for this new magic square? If so, write out the new system. If not, explain why.
Hint: Think about what you did in the previous part. In combination with properties of e, can you transform this new system into a linear form?
Solution:
Let's consider the first row, with the equation:

$$e^{x_{11}} \times e^{x_{12}} \times e^{x_{13}} = k$$

Noting that the product of exponentials allows you to sum the exponents, this is equivalent to:

$$e^{x_{11}+x_{12}+x_{13}} = k$$

Applying the natural log results in the following linear equation:

$$x_{11} + x_{12} + x_{13} = ln(k)$$

$$\begin{aligned}
x_{11} + x_{12} + x_{13} &= ln(k) \\
x_{21} + x_{22} + x_{23} &= ln(k) \\
x_{31} + x_{32} + x_{33} &= ln(k) \\
x_{11} + x_{21} + x_{31} &= ln(k) \\
x_{12} + x_{22} + x_{32} &= ln(k) \\
x_{13} + x_{23} + x_{33} &= ln(k) \\
x_{11} + x_{22} + x_{33} &= ln(k) \\
x_{31} + x_{22} + x_{13} &= ln(k)
\end{aligned}$$

4. Recognizing Linear Equations

Your TA, Aniruddh, has recently started taking EECS C106A (Robotics), and is brushing up on his physics knowledge. He remembers the following formula describing the position of an object with respect to time:

$$x = v_0 t + \frac{at^2}{2}$$

Here, we assume a starting position of 0 meters, where v_0 represents the initial velocity, and a represents the acceleration (assumed to be constant).

(a) A lot of robotics involves understanding system parameters based on measurements. You consider thinking about this from a 16A lens, and first want to see if the equation is linear. Is the equation linear with respect to t? In other words, is the function $x(t)$ linear? How about with respect to v_0 AND a (i.e is $x(v_0, a)$ linear)? If it is linear, show the properties of homogeneity and superposition hold for those variables. If not, explain which property it violates.

Solution: This is NOT linear with respect to t, as we have a squared term. Notice, however, that it IS linear with respect to a, and v_0. Once the t and x values are set, we end up with an equation that looks like this:

$$c_1 = c_2 v_0 + c_3 a$$

which follows all the linear properties, for c_i that are constant, real numbers.

Homogeneity:

$$f(\alpha v_0, \alpha a) = \alpha(c_2 v_0 + c_3 a) = \alpha f(v_0, a)$$

Superposition:

$$f(v_{0,1} + v_{0,2}, a_1 + a_2) = c_2(v_{0,1} + v_{0,2}) + c_3(a_1 + a_2) = f(v_{0,1}, a_1) + f(v_{0,2}, a_2)$$

(b) You decide to test your theory of linearity by taking measurements of a projectile thrown by the robot arm. You record the following measurements:

- At t = 1 second, the position, x, is measured to be 1 meter.
- At t = 2 seconds, the position, x, is measured to be -7.8 meters.

Can you set this up as a system of linear equations and calculate the value of v_0 and a?
Solution: The equations are as follows:

$$1v_0 + 0.5a = 1$$

$$2v_0 + 2a = -7.8$$

Solving this simple system yields:

$$v_0 = 5.9m/s$$

$$a = -9.8m/s^2$$

5. Word Problems

Learning Objective: *Understand how to setup a system of linear equations from word problems.*

For these word problems, represent the system of linear equations as an augmented matrix. Then, solve the system using substitution or Gaussian elimination.

(a) Gustav is collecting soil samples. Each soil sample contains some sand, some clay, and some organic material. He wants to know the density of each material. His first sample has 0.5 liters of sand, 0.25 liters of clay, and 0.25 liters of organic material, and weighs 1.625 kg. His second sample contains 1 liter of sand, 0 liters of clay, and 1 liter of organic material, and weighs 3 kg. His third sample contains 0.25 liters of sand, 0.5 liters of clay, and 0 liters of organic material, and weighs 1.375 kg Let s be the density of sand, c be the density of clay, and m be the density of organic material, all measured in kg/L. Solve for the density of each material.

Solution: We translate the system of equations to an augmented matrix and solve.

$$\left[\begin{array}{ccc|c} 0.5 & 0.25 & 0.25 & 1.625 \\ 1 & 0 & 1 & 3 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right]$$

To solve, we use substitution. Rearranging the second equation,

$$s = 3 - m$$

Rearranging the third equation,

$$0.5c = 1.375 - 0.25s$$

Plugging in $s = 3 - m$ into this new equation and multiplying both sides by 2, we find that

$$c = 2.75 - 0.5(3 - m)$$

Now that we have an expression for s and c in terms of m, we can plug these expressions into the first equation to solve for m. This yields

$$0.5(3 - m) + 0.25(2.75 - 0.5(3 - m)) + 0.25m = 1.625$$

$$m = 1.5$$

Backsubstituting into the second equation yields

$$s = 1.5$$

And again into the third equation

$$c = 2$$

We find sand has a density of 1.5 kg/L, clay has a density of 2 kg/L, and organic material has a density of 1.5 kg/L.

Alternatively, using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 0.5 & 0.25 & 0.25 & 1.625 \\ 1 & 0 & 1 & 3 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 1 & 0 & 1 & 3 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right] \text{using } 2R_1 \rightarrow R_1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 0.25 & 0.5 & 0 & 1.375 \end{array}\right] \text{using } R_2 - R_1 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 1 & 2 & 0 & 5.5 \end{array}\right] \text{using } 4R_3 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 0 & 1.5 & -0.5 & 2.25 \end{array}\right] \text{using } R_3 - R_1 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0.5 & -0.25 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_3 + 3R_2 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & -0.5 & 0 & -1 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_2 - \tfrac{1}{2}R_3 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0.5 & 0.5 & 3.25 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } -2R_2 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1.5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1.5 \end{array}\right] \text{using } R_1 - \tfrac{1}{2}R_2 - \tfrac{1}{2}R_3 \rightarrow R_1$$

(b) Alice buys 3 apples and 4 oranges for 17 dollars. Bob buys 1 apple and 10 oranges for 23 dollars (Bob really likes oranges). How much do apples and oranges cost individually?

Solution: Treat x as the cost of one apple and y as the cost of one orange.

$$\left[\begin{array}{cc|c} 3 & 4 & 17 \\ 1 & 10 & 23 \end{array}\right]$$

Using substitution, we first isolate x from the first equation.

$$x = \frac{17}{3} - \frac{4}{3}y$$

Plugging this into the second equation, we find

$$\frac{17}{3} - \frac{4}{3}y + 10y = 23$$

Solving for y yields

$$y = 2$$

Backsubstition into the first equation yields

$$3x + 4 \times 2 = 17$$

$$x = 3$$

Alternatively, using Gaussian elimination:

$$\left[\begin{array}{cc|c} 3 & 4 & 17 \\ 1 & 10 & 23 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{17}{3} \\ 1 & 10 & 23 \end{array}\right] \text{using } \tfrac{1}{3}R_1 \to R_1$$
$$\to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{17}{3} \\ 0 & \frac{26}{3} & \frac{52}{3} \end{array}\right] \text{using } R_2 - R_1 \to R_2$$
$$\to \left[\begin{array}{cc|c} 1 & \frac{4}{3} & \frac{17}{3} \\ 0 & 1 & 2 \end{array}\right] \text{using } \tfrac{3}{26}R_2 \to R_2$$
$$\to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \text{using } R_1 - \tfrac{4}{3}R_2 \to R_1$$

We find apples cost \$3 each and oranges cost \$2 each.

(c) Jack, Jill, and James are driving from Berkeley to Las Vegas. Each of them takes a different route. Jack takes a short route and ends up going through Toll Road A and Toll Road B, costing him \$10. Jill takes a slightly longer route and goes through Toll Road B and Toll Road C, costing her \$15. Finally, James takes a wrong turn and takes Toll Road A twice, then takes Toll Road B and finally Toll Road C, costing him \$25. What is the toll cost on each road?

Solution: Treat a, b, c as the toll cost on each of Toll Roads A, B, and C respectively.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 2 & 1 & 1 & 25 \end{array}\right]$$

Solving with substitution, we start with the second equation and solve for b

$$b = 15 - c$$

Rearranging the first equation and plugging in our previous results yields

$$a = 10 - b = 10 - (15 - c) = c - 5$$

Plugging both of these equations into the last equation, we find

$$2(c-5) + (15-c) + c = 25$$

$$c = 10$$

Backsubstition yields

$$a = 5$$

$$b = 5$$

Alternatively, using Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 2 & 1 & 1 & 25 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 0 & -1 & 1 & 5 \end{array}\right] \text{using } R_3 - 2R_1 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 2 & 20 \end{array}\right] \text{using } R_3 + R_2 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{using } \tfrac{1}{2}R_3 \rightarrow R_3$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 10 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{using } R_2 - R_3 \rightarrow R_2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 10 \end{array}\right] \text{using } R_1 - R_2 \rightarrow R_1$$

We find Toll Road A cost \$5, Toll Road B cost \$5, and Toll Road C cost \$10.

6. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Please remember to submit both your homework as well as the self-grade assignment following the release of the solutions. A full description of the submission process is listed on the class website (eecs16a.org).

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.