EECS 16A Designing Information Devices and Systems I
Fall 2022 Discussion 12B

1. Inner Product Properties

For this question, we will verify our definition of the Euclidean inner product in Cartesian coordinates

$$\langle \vec{x}, \vec{y} \rangle \;=\; x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \;, \quad \text{for any } \; \vec{x}, \vec{y} \in \mathbb{R}^n$$

indeed satisfies the key properties required for all inner products for the 2-dimensional case. Suppose $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^2$ for the following parts:

(a) Show symmetry: $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$.

Answer: This is seen by direct expansion:
Let $x_i, y_i \in \mathbb{R}$, then

$$\begin{aligned}
\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle &= x_1 \cdot y_1 + x_2 \cdot y_2 \\
&= y_1 \cdot x_1 + y_2 \cdot x_2 \\
&= \left\langle \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle
\end{aligned}$$

(b) Show linearity: $\langle \vec{x}, c\vec{y} + d\vec{z} \rangle = c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle$, where $c, d \in \mathbb{R}$ are real numbers.

Answer: This is accomplished through a direct expansion:

$$\begin{aligned}
\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, c\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + d\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} cy_1 + dz_1 \\ cy_2 + dz_2 \end{bmatrix} \right\rangle \\
&= x_1(cy_1 + dz_1) + x_2(cy_2 + dz_2) \\
&= c(x_1 y_1 + x_2 y_2) + d(x_1 z_1 + x_2 z_2) \\
&= c\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right\rangle + d\left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right\rangle \\
&= c\langle \vec{x}, \vec{y} \rangle + d\langle \vec{x}, \vec{z} \rangle
\end{aligned}$$

(c) Show non-negativity: $\langle \vec{x}, \vec{x} \rangle \geq 0$, with equality if and only if $\vec{x} = \vec{0}$.

Answer: This part requires just a bit more thought beyond a direct expansion of $\langle \vec{x}, \vec{x} \rangle$, but we first recognize that this inner product is the definition of the norm (or length) of $\vec{x}$. So it is at least in intuitive that a length of some vector (squared) cannot be negative:

$$\begin{aligned} \langle \vec{x}, \vec{x} \rangle &= \left\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\rangle \\ &= x_1^2 + x_2^2 \end{aligned}$$

From this result we notice if either x_1 or x_2 are nonzero (even negative) values, then the inner product HAS to be positive. The only case in which the inner product $\langle \vec{x}, \vec{x} \rangle$ is identically zero is when both $x_1 = 0$ AND $x_2 = 0$, which verifies the final part of the property: $\langle \vec{x}, \vec{x} \rangle = 0$ ONLY IF $\vec{x} = \vec{0}$.

As a bonus, suppose we re-label our vector components $x_1 = a$ and $x_2 = b$.
The we see $\langle \vec{x}, \vec{x} \rangle = c^2 = a^2 + b^2$, which is the Pythagorean theorem!
This verifies that $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = c$ can be geometrically understood as the length of vector $\vec{x}$.

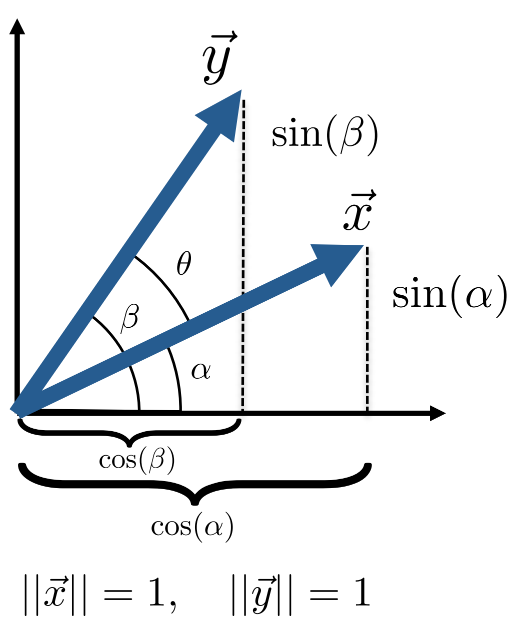
2. Geometric Interpretation of the Inner Product

In this problem, we explore the geometric interpretation of the Euclidean inner product, restricting ourselves to vectors in $\mathbb{R}^2$.

Remember that the formula for the inner product of two vectors can be expressed in terms of their magnitudes and the angle between them as follows:

$$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cdot \cos\theta$$

The figure below may be helpful in illustrating this property:



For each subpart, give an example of any two (nonzero) vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ that satisfy the stated condition and compute their inner product.

(a) Give an example of a pair of parallel vectors (vectors that point in the same direction and have an angle of 0 degrees between them).

Answer: Parallel vectors point in the same direction (have an angle of 0° between them).
This means we must have $\vec{y} = \alpha\vec{x}$ for some $\alpha > 0$.
Having only this condition leaves a lot of freedom.
Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{y} = 2\,\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot 2 + 1 \cdot 2 = 4$$

(b) Give an example of a pair of anti-parallel vectors (vectors that point in opposite directions).

Answer: Anti-parallel vectors point in opposite directions (have an angle of 180° between them).
This means we must have $\vec{y} = \alpha\vec{x}$ again, but now for some negative $\alpha < 0$.
Having only this condition still leaves a lot of freedom.

Let us choose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and then set $\vec{y} = -2\,\vec{x} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot -2 + 1 \cdot -2 = -4$$

(c) Give an example of a pair of perpendicular vectors (vectors that have an angle of 90 degrees between them).

Answer: Perpendicular vectors point in 90° directions with respect to each-other.
Most importantly, the Euclidean inner product $\langle \vec{x}, \vec{y} \rangle = 0$ whenever $\vec{x}, \vec{y}$ are orthogonal, or perpendicular.

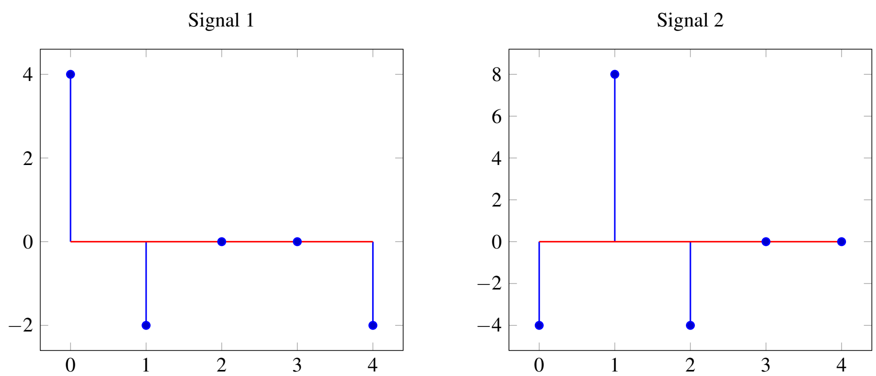
For our example we will fix $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and then leave $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ general.

$$\langle \vec{x}, \vec{y} \rangle = 1 \cdot y_1 + 0 \cdot y_2 = y_1 \ \equiv 0.$$

Thus we must set $y_1 = 0$, but y_2 can assume any nonzero value!

3. Correlation

(a) You are given the following two signals:



Sketch the linear cross-correlation of signal 1 with signal 2. That is, find: $\text{corr}(\vec{s}_1, \vec{s}_2)[n]$ for $n = 0, 1, .., 4$. Do not assume the signals are periodic.

Answer:

Represent signal 1 as the vector $\vec{s}_1 = \begin{bmatrix} 4 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, zero-padded so that we compute only the linear correlation. Similarly, represent signal 2 as the vector $\vec{s}_2 = \begin{bmatrix} -4 & 8 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, where we once again zero pad the vector. Notice that we zero pad the vectors $\vec{s}_1$ and $\vec{s}_2$ to represent the signals from $n = 0, 1, ..., 8$. This is because we are only interested in calculating the cross-correlation for for $n = 0, 1, .., 4$, therefore we will only need to shift the vector $\vec{s}_2$ four times.

The cross-correlation between two vectors is defined as follows:

$$\text{corr}(\vec{x}, \vec{y})[k] = \sum_{i=-\infty}^{\infty} \vec{x}[i]\vec{y}[i-k]$$

To compute the cross-correlation $\text{corr}(\vec{s}_1, \vec{s}_2)$, we shift the vector $\vec{s}_2$ and compute the inner product of the shifted $\vec{s}_2$ and the vector $\vec{s}_1$.

$\vec{s}_1$	4		-2		0		0		-2		0		0		0		0	
$\vec{s}_2[n]$	-4		8		-4		0		0		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	-16	+	-16	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= -32

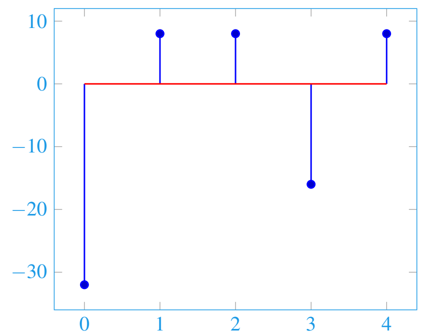
$\vec{s}_1$	4		-2		0		0		-2		0		0		0		0	
$\vec{s}_2[n-1]$	0		-4		-8		-4		0		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0	+	8	+	0	+	0	+	0		0	+	0	+	0	+	0	= 8

$\vec{s}_1$	4		-2		0		0		-2		0		0		0		0	
$\vec{s}_2[n-2]$	0		0		-4		8		-4		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	8	+	0	+	0	+	0	+	0	= 8

$\vec{s}_1$	4		-2		0		0		-2		0		0		0		0	
$\vec{s}_2[n-3]$	0		0		0		-4		8		-4		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n-3] \rangle$	0	+	0	+	0	+	0	+	-16	+	0	+	0	+	0	+	0	= -16

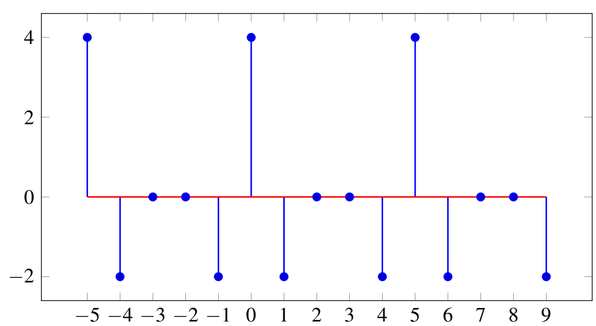
$\vec{s}_1$	4	-2	0	0	-2	0	0	0	0	
$\vec{s}_2[n-4]$	0	0	0	0	-4	8	-4	0	0	
$\langle \vec{s}_1, \vec{s}_2[n-4] \rangle$	0 +	0 +	0 +	0 +	8 +	0 +	0 +	0 +	0	= 8

Non-periodic Cross-correlation of Signals 1 and 2

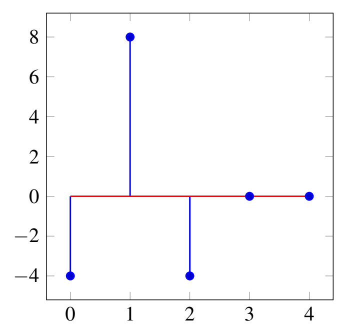


(b) Now, the pattern in $\vec{s}_1$ is repeated three times:

Signal 1



Signal 2



Sketch the linear cross-correlation of signal 1 with signal 2, $\text{corr}(\vec{s}_1, \vec{s}_2)[n]$, for $n = 0, 1, .., 4$.

Answer: Recall that $\text{corr}(\vec{x}, \vec{y})[k] = \sum_{i=-\infty}^{\infty} \vec{x}[i]\vec{y}[i-k]$

As we did in part a) to compute the cross-correlation $\text{corr}(\vec{s}_1, \vec{s}_2)$, we shift the vector $\vec{s}_2$ and compute the inner product of the shifted $\vec{s}_2$ and the vector $\vec{s}_1$. Since we are interested in $\text{corr}(\vec{s}_1, \vec{s}_2)[n]$, for $n = 0, 1, .., 4$, here we have shown the two signals for $n = 0, 1, .., 8$.

$\vec{s}_1$	4	-2	0	0	-2	4	-2	0	0	-2	
$\vec{s}_2[n]$	-4	8	-4	0	0	0	0	0	0	0	
$\langle \vec{s}_2, \vec{s}_1[n] \rangle$	-16 +	-16 +	0 +	0 +	0 +	0 +	0 +	0 +	0 +	0	= -32

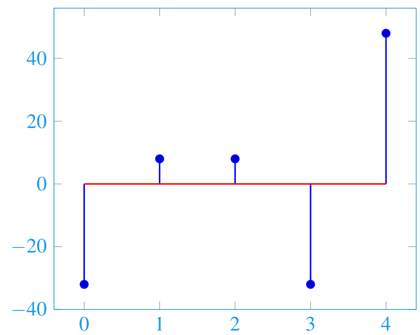
$\vec{s}_1$	4	-2	0	0	-2	4	-2	0	0	-2	
$\vec{s}_2[n-1]$	0	-4	8	-4	0	0	0	0	0	0	
$\langle \vec{s}_2, \vec{s}_1[n-1] \rangle$	0 +	8 +	0 +	0 +	0 +	0 +	0 +	0 +	0 +	0	= 8

$\vec{s}_1$	4	-2	0	0	-2	4	-2	0	0	-2	
$\vec{s}_2[n-2]$	0	0	-4	8	-4	0	0	0	0	0	
$\langle \vec{s}_2, \vec{s}_1[n-2] \rangle$	0 +	0 +	0 +	0 +	8 +	0 +	0 +	0 +	0 +	0	= 8

$\vec{s}_1$	4	-2	0	0	-2	4	-2	0	0	-2	
$\vec{s}_2[n-3]$	0	0	0	-4	8	-4	0	0	0	0	
$\langle \vec{s}_2, \vec{s}_1[n-3] \rangle$	0 +	0 +	0 +	0 +	-16 +	-16 +	0 +	0 +	0 +	0	= -32

$\vec{s}_1$	4	-2	0	0	-2	4	-2	0	0	-2	
$\vec{s}_2[n-4]$	0	0	0	0	-4	8	-4	0	0	0	
$\langle \vec{s}_2, \vec{s}_1[n-4] \rangle$	0 +	0 +	0 +	0 +	8 +	32 +	8 +	0 +	0 +	0	$= 48$

Cross-correlation of Signals 1 and 2



Notice that when $\vec{s}_1$ is periodic we don't simply get the result from part a) repeated.