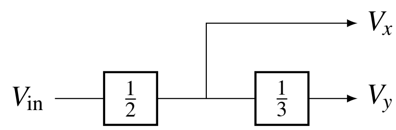
EECS 16A Designing Information Devices and Systems I
Fall 2022 Discussion 11A

1. **Modular Circuit Buffer**

Let's try designing circuits that perform a set of mathematical operations using op-amps. While voltage dividers on their own cannot be combined without altering their behavior, op-amps can preserve their behavior when combined and thus are a perfect tool for modular circuit design. We would like to implement the block diagram shown below:



In other words, create a circuit with two outputs V_x and V_y, where $V_x = \frac{1}{2}V_{\text{in}}$ and $V_y = \frac{1}{3}V_x = \frac{1}{6}V_{\text{in}}$.

(a) Draw two voltage dividers, one for each operation (the $1/2$ and $1/3$ scalings). What relationships hold for the resistor values for the $1/2$ divider, and for the resistor values for the $1/3$ divider?

Answer: Recall our voltage divider consists of V_{in} connected to two resistors (R_1, R_2) in series with the output voltage between ground and the central node. This yields the formula

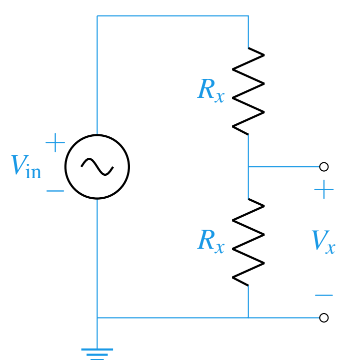
$$V_{\text{out}} = \left(\frac{R_2}{R_1 + R_2}\right) V_{\text{in}}.$$

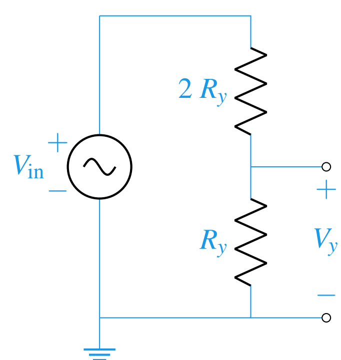
For the $1/2$ operation (V_x output) we recognize

$$\frac{1}{2} = \left(\frac{R_2}{R_1 + R_2}\right) \quad \longrightarrow \quad R_1 + R_2 = 2R_2 \quad \longrightarrow \quad R_1 = R_2 \equiv R_x.$$

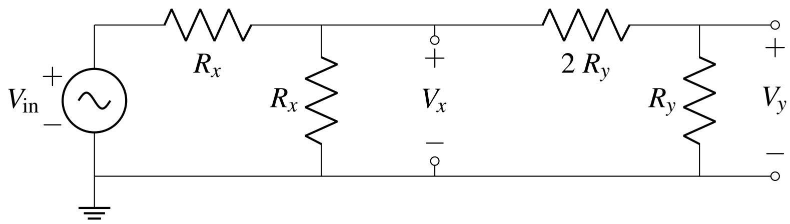
For the $1/3$ operation (V_y output) we recognize

$$\frac{1}{3} = \left(\frac{R_2}{R_1 + R_2}\right) \quad \longrightarrow \quad R_1 + R_2 = 3R_2 \quad \longrightarrow \quad \frac{R_1}{2} = R_2 \equiv R_y.$$





(b) If you combine the voltage dividers, made in part (a), as shown by the block diagram (output of the $1/2$ voltage divider becomes the source for the $1/3$ voltage divider circuit), do they behave as we hope (meaning $V_{\text{in}} = 2V_x = 6V_y$)?



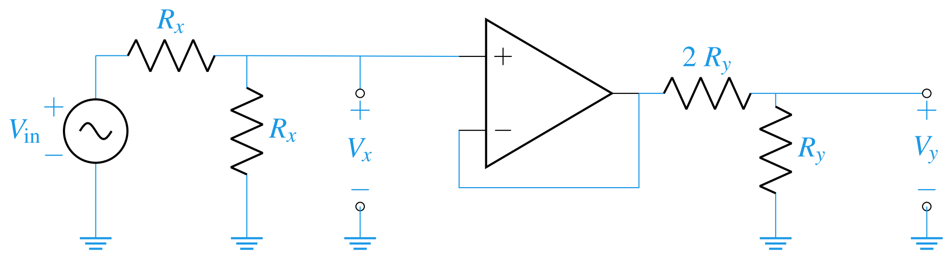
Answer: To quickly access this combined system, we may identify V_x as the result of a new equivalent voltage divider (recognizing the R_y resistors in series and that series is in parallel with R_x). The load resistor becomes $R_{eq} = \frac{3R_xR_y}{R_x+3R_y}$. This yields

$$V_x = \left(\frac{R_{eq}}{R_x + R_{eq}}\right) V_{\text{in}} = \left(\frac{1}{2 + \frac{R_x}{3R_y}}\right) V_{\text{in}} \qquad V_y = \frac{1}{3} V_x = \left(\frac{1}{6 + \frac{R_x}{R_y}}\right) V_{\text{in}}$$

From this stage it is evident that combining our dividers changes their behavior (although they preserve behavior in the limit $R_y >> R_x$).
The new values for V_x, V_y are dependent on values from both dividers, which means they can't be treated independently! $\square$.

(c) Perhaps we could use an op-amp (in negative-feedback) to achieve our desired behavior.
Modify the implementation you tried in part (b) using a negative feedback op-amp in order to achieve the desired V_x, V_y relations $V_x = \frac{V_{\text{in}}}{2}$ and $V_y = \frac{V_x}{3} = \frac{V_{\text{in}}}{6}$.
HINT: Place the op-amp in between the dividers such that the V_x node is an input into the op-amp, while the source of the 2nd divider is the output of the op-amp!

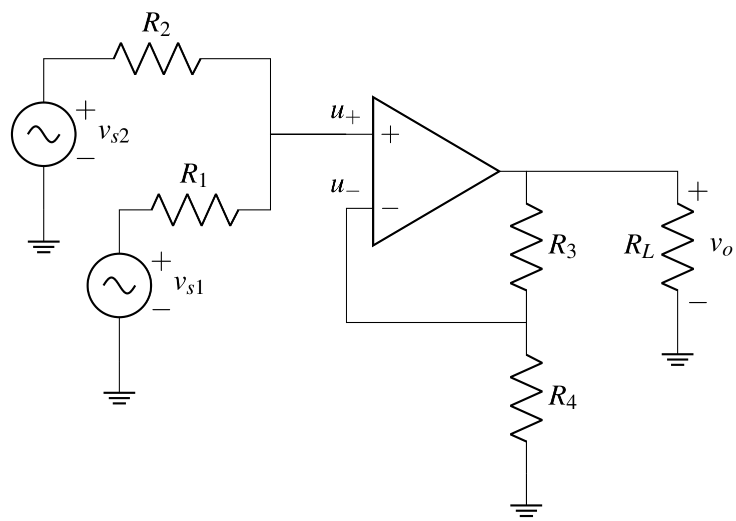
Answer: Use the op-amp as a voltage buffer.
This means we short the op-amp's negative input to its output, since the positive input must now match its output (by the golden rules).



Since no current flows into the positive op-amp input, we've successfully isolated the dividers so they can be used in a modular fashion! □

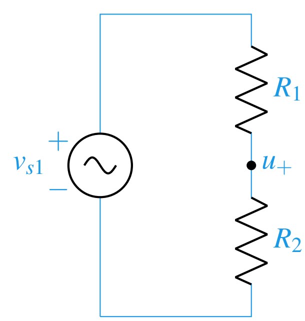
NOTE: The V_x, V_y outputs from this configuration would change with the addition of a load on either terminal. As a follow-up, think about ways to make each output agnostic to the loads attached!

2. Multiple Inputs To One Op-Amp



(a) First, let's focus on the left part of the circuit containing the voltage sources v_{s1} and v_{s2}, and resistances R_1 and R_2. Solve for u_+ in the circuit above. *(Hint: Use superposition.)*

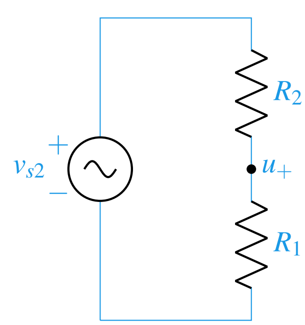
Answer: Let's call the potential at the positive input of the op-amp u_+. Using superposition, we first turn off v_{s2} and find u_+. The circuit then looks like:



We recognize the above circuit as a voltage divider. Thus,

$$u_{+,vs1} = \frac{R_2}{R_1 + R_2} v_{s1}$$

By symmetry, we expect v_{s2} to have a similar circuit and expression. The circuit for v_{s2} looks like:



The expression for u_+ with v_{s2} is then:

$$u_{+,vs2} = \frac{R_1}{R_1 + R_2} v_{s2}$$

From superposition, we know the output must be the sum of these.

$$u_+ = \frac{R_2}{R_1 + R_2} v_{s1} + \frac{R_1}{R_1 + R_2} v_{s2}$$

(b) How would you choose R_1 and R_2 that produce a voltage $u_+ = \frac{1}{2}V_{s1} + \frac{1}{2}V_{s2}$? Could you also achieve $u_+ = \frac{1}{3}V_{s1} + \frac{2}{3}V_{s2}$

Answer:

We found that the output voltage for any two resistors R_1 and R_2 is:

$$v_+ = \frac{R_2}{R_1 + R_2}V_1 + \frac{R_1}{R_1 + R_2}V_2$$

Thus, to create the $\frac{1}{2}$ - $\frac{1}{2}$ ratio, we can use any nonzero resistances with value R such that:

$$R_1 = R \qquad R_2 = R$$

Similarly, to create the $\frac{1}{3}$ - $\frac{2}{3}$ ratio, we can use:

$$R_1 = 2R \qquad R_2 = R$$

In general, you can achieve anything of the form $u_+ = kV_1 + (1-k)V_1$ with $k \in (0,1)$! This is a very useful topology to combine two voltages.

(c) Now, for the whole circuit, find an expression for v_o.

Answer:

With u_+ determined, we can find the output voltage directly from the formula for a non-inverting amplifier. We can also derive it using the process below.

From the negative feedback rule, $u_+ = u_-$. Using voltage dividers, we can express u_- in terms of v_o:

$$u_- = \frac{R_4}{R_3 + R_4}v_o$$

$$v_o = \left(1 + \frac{R_3}{R_4}\right)u_- = \left(1 + \frac{R_3}{R_4}\right)u_+$$

Now, to find the final output, we can set u_+ to our earlier expression.

$$v_o = \left(1 + \frac{R_3}{R_4}\right)\left(\frac{R_2}{R_1 + R_2}v_{s1} + \frac{R_1}{R_1 + R_2}v_{s2}\right)$$

(d) How could you use this circuit to find the sum of different signals, i.e. $V_{s1} + V_{s2}$? What about taking the sum and multiplying by 2, i.e. $2(V_{s1} + V_{s2})$?

Answer:

The circuit already finds the weighted sum of two inputs. By setting $R_1 = R_2$ and $R_3 = R_4$, we can take the exact sum of two inputs.

$$v_o = \left(1 + \frac{R_3}{R_4}\right)\left(\frac{R_2}{R_1 + R_2} v_{s1} + \frac{R_1}{R_1 + R_2} v_{s2}\right) = (1+1)\left(\frac{1}{2} v_{s1} + \frac{1}{2} v_{s2}\right) = v_{s1} + v_{s2}$$

Notice that the first half of this circuit (R_1 and R_2) form a voltage summer with coefficients less than one; the second half is just a non-inverting amplifier. Thus we can always use R_1 and R_2 to take an equally weighted sum of the inputs and then multiply greater than 1 using the non-inverting amplifier. If we set $R_1 = R_2$, we get $\left(\frac{1}{2} v_{s1} + \frac{1}{2} v_{s2}\right)$ into the op-amp. To get a total gain of 2, then the non-inverting op-amp needs a gain of 4, so we can pick $R_3 = 3R_4$.

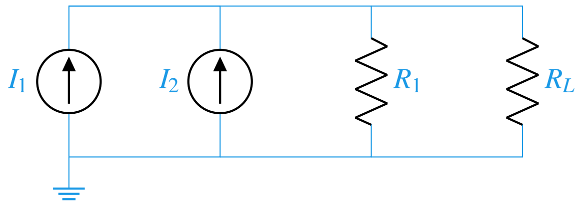
3. (Optional) Designing a Current Divider

(a) You have two current sources, I_1 and I_2. You also have a load resistor $R_L = 6\,\mathrm{k}\Omega$. You can use whatever resistors you want (as long as they are finite integer multiples of $1\,\mathrm{k}\Omega$). How would you design a circuit such that the current running through R_L is $I_L = \frac{2}{5}(I_1 + I_2)$?

Answer:

Use superposition, so think of the two currents as one summed current. Then, use KCL to determine how to divide the currents. Remember, the current divider formula is similar to that of the voltage divider, with the numerators flipped. This means that in the current divider, when calculating the current through one resistor we place the other resistor in the numerator, i.e.:

$$I_{R1} = \frac{R_L}{R_1 + R_L}(I_1 + I_2), \quad I_{R_L} = \frac{R_1}{R_1 + R_L}(I_1 + I_2)$$



Using the above equations, we get that one possible resistor combination that creates $I_L = \frac{2}{5}(I_1 + I_2)$ is:

$$R_L = 6\,\mathrm{k}\Omega, R_1 = 4\,\mathrm{k}\Omega$$