# EECS 16A Designing Information Devices and Systems I
# Fall 2022 Discussion 5B

## 1. Mechanical Determinants

(a) Compute the determinant of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

**Answer:**

We can use the form of a $2 \times 2$ determinant from lecture:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

Therefore,

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\right) = 2 \cdot 3 - 0 \cdot 0 = 6$$

(b) Compute the determinant of $\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}$.

**Answer:**

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a \cdot \det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \cdot \det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c \cdot \det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

Therefore,

$$\begin{aligned} \det\left(\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix}\right) &= 2 \cdot \det\left(\begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix}\right) + 3 \cdot \det\left(\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}\right) + 1 \cdot \det\left(\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}\right) \\ &= 2 \cdot (0 - (-4)) + 3 \cdot (10 - (-1)) + 1 \cdot (8 - 0)) \\ &= 8 + 33 + 8 \\ &= 49 \end{aligned}$$

## 2. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and their associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Do you observe anything about the eigenvalues and eigenvectors?

**Answer:**

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) = \det\left(\begin{bmatrix}1-\lambda & 0\\0 & 9-\lambda\end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.
For the eigenvalue $\lambda = 1$:

$$\begin{aligned}(\mathbf{M} - 1\mathbf{I})\vec{x} &= \vec{0}\\ \left(\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} - 1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\vec{x} &= \vec{0}\\ \begin{bmatrix}0 & 0\\0 & 8\end{bmatrix}\vec{x} &= \vec{0}\end{aligned}$$

From the second equation in the system, $x_2 = 0$, with any solution having the form $\begin{bmatrix}1\\0\end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is thus span $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.
For the eigenvalue $\lambda = 9$:

$$\begin{aligned}(\mathbf{M} - 9\mathbf{I})\vec{x} &= \vec{0}\\ \left(\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} - 9\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\vec{x} &= \vec{0}\\ \left(\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} - \begin{bmatrix}9 & 0\\0 & 9\end{bmatrix}\right)\vec{x} &= \vec{0}\\ \begin{bmatrix}-8 & 0\\0 & 0\end{bmatrix}\vec{x} &= \vec{0}\end{aligned}$$

From the first equation in the system, $x_1 = 0$, so any solution must take the form $\begin{bmatrix}0\\1\end{bmatrix} t$ for $t \in \mathbb{R}$. The eigenspace is span $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$.
We observe that the eigenvalues are just the diagonal entries. Since the matrix is diagonal, multiplying the diagonal matrix $\mathbf{D}$ with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries $d_i$ of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(b) $\mathbf{M} = \begin{bmatrix}0 & 1\\-2 & -3\end{bmatrix}$

**Answer:**

Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix}0-\lambda & 1\\-2 & -3-\lambda\end{bmatrix}\right) = 0$$

$$
\begin{aligned}
-\lambda(-3-\lambda)+2 &= 0 \\
\lambda^2+3\lambda+2 &= 0 \\
(\lambda+2)(\lambda+1) &= 0 \\
\lambda &= -1,-2
\end{aligned}
$$

$\lambda=-1$:

$$
\left[\begin{array}{cc|c} 0-(-1) & 1 & 0 \\ -2 & -3-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

$$
\begin{aligned} x_1+x_2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t
$$

The eigenspace for $\lambda=-1$ is $\operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

$\lambda=-2$:

$$
\left[\begin{array}{cc|c} 0-(-2) & 1 & 0 \\ -2 & -3-(-2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]
$$

$$
\begin{aligned} x_1+x_2/2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t/2 \\ t \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t
$$

The eigenspace for $\lambda=-2$ is $\operatorname{span}\left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}$.

## 3. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where $\lambda$ is a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

**Answer:** Multiplying the identity matrix with any vector in $\mathbb{R}^n$ produces the same vector, that is, $\mathbf{I}\vec{x} = \vec{x} = 1\cdot\vec{x}$. Therefore, $\lambda = 1$. Since $\vec{x}$ can be any vector in $\mathbb{R}^n$, the corresponding eigenvectors are all vectors in $\mathbb{R}^n$.

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

**Answer:** Since the matrix is diagonal, multiplying the diagonal matrix with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries $d_i$ of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(c) Conceptually, does a rotation matrix in $\mathbb{R}^2$ by angle $\theta$ have any eigenvalues $\lambda \in \mathbb{R}$? For which angles is this the case?

**Answer:** In a conceptual sense, there are three cases:

**Rotation by $0^\circ$:** (more accurately, any integer multiple of $360^\circ$), which yields a rotation matrix $\mathbf{R} = \mathbf{I}$: This will have one eigenvalue of $+1$ because it doesn't affect any vector ($\mathbf{R}\vec{x} = \vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

**Rotation by $180^\circ$:** (more accurately, any angle of $180^\circ + n \cdot 360^\circ$ for integer $n$), which yields a rotation matrix $\mathbf{R} = -\mathbf{I}$: This will have one eigenvalue of $-1$ because it "flips" any vector ($\mathbf{R}\vec{x} = -\vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

**Any other rotation:** there aren't any real eigenvalues. The reason is, if there were any real eigenvalue $\lambda \in \mathbb{R}$ for a non-trivial rotation matrix, it means that we can get $\mathbf{R}\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$, which means that by rotating a vector, we scaled it. This is a contradiction (again, unless $\mathbf{R} = \mathbf{I}$). Refer to Figure 1 for a visualization.

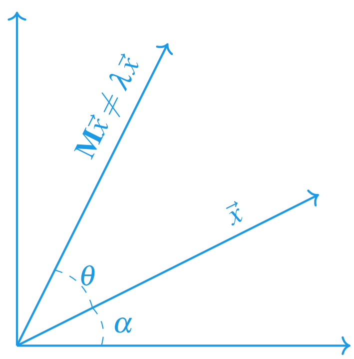


Figure 1: Rotation will never scale any non-zero vector (by a real number) unless it is rotation by an integer multiple of $360^\circ$ (identity matrix) or the rotation angle is $\theta = 180^\circ + n \cdot 360^\circ$ for any integer $n$ ($-\mathbf{I}$).

(d) **(PRACTICE)** Now let us mechanically compute the eigenvalues of the rotation matrix in $\mathbb{R}^2$. Does it agree with our findings above? As a refresher, the rotation matrix $\mathbf{R}$ has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

**Answer:** Using our known determinant formula for 2x2 matrices $\det(A) = ad - bc$ we can compute the characteristic polynomial

$$\det(\mathbf{R} - \lambda\mathbf{I}) = \det \begin{bmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{bmatrix} = \cos(\theta)^2 + \sin(\theta)^2 - 2\cos(\theta)\lambda + \lambda^2 \equiv 0$$

From here we can first simplify $1 = \cos(\theta)^2 + \sin(\theta)^2$ and then use the quadratic formula to attain the two possible $\lambda$ values.

$$\lambda = \cos(\theta) \pm \sqrt{\cos(\theta)^2 - 1} = \cos(\theta) \pm i\sqrt{1 - \cos(\theta)^2} = \cos(\theta) \pm i\sqrt{\sin(\theta)^2}$$

In exponential phase notation we can write the two eigenvalues more concisely: $\lambda = e^{\pm i\theta}$

(e) Does the reflection matrix $\mathbf{T}$ across the x-axis in $\mathbb{R}^{2\times 2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Answer:** Yes, both $+1$ and $-1$. Mechanically, we could go through the methods we have learned for attaining a characteristic polynomial from $\det(T-\lambda I) = (1-\lambda)(-1-\lambda)-(0)(0)$ and recalling our eigenvalues are the roots of this polynomial (the values where this polynomial is zero). This works because matrix $T-\lambda I$ only has a nonempty null space when its determinant is zero!

$$\det(T-\lambda I) = \lambda^2 - 1 \equiv 0 \quad \rightarrow \quad \lambda = \pm 1$$

Conceptually, we can reason that a vector along the x-axis will be unaffected by $\mathbf{T}$ (in this case $\lambda = +1$), where as a vector along the y-axis gets perfectly flipped by $\mathbf{T}$ (in this case $\lambda = -1$)

NOTE: A $2\times 2$ reflection matrix always has $\lambda = \pm 1$, REGARDLESS of the axis of reflection. Why? Reflecting any vector that is on the reflection axis will not affect it (eigenvalue $+1$). Reflecting any vector orthogonal (perpendicular) to the reflection axis will just "flip it/negate it" (eigenvalue $-1$). In other words, the set of vectors that lie along the axis of reflection is the eigenspace associated with the eigenvalue $+1$ and the set of vectors orthogonal to the axis of reflection is the eigenspace associated with the eigenvalue $-1$.

(f) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

**Answer:** $N(A)$ is not just $\vec{0}$ as we have some $\vec{v} \neq \vec{0}$ satisfying $A\vec{v} = \lambda\vec{v}$. Another way we can state this is that $\dim(N(A)) > 0$.
Thus we can imagine if $\mathbf{M}\vec{x} = \vec{b}$ has a solution then $\mathbf{M}(\vec{x}+\vec{v}) = \vec{b}$ also solves the system, hence there are infinite solutions. Yet we also know that a nonzero null space means $\mathbf{M}$ has linearly dependent columns, so the vector $\vec{b}$ could lie outside of this span in which case there is no solution.

In summary, there are either infinite or no solutions to the system of equations $\mathbf{M}\vec{x} = \vec{b}$

(g) **(Practice)** Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

**Answer:**

Note that the matrix has linearly dependent columns. Therefore, according to part (f), one eigenvalue is $\lambda = 0$. The corresponding eigenvector, which is equivalent to the basis vector for the null space, is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The other eigenvalue is, by inspection, $\lambda = 1$ with the corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.