EECS 16A Designing Information Devices and Systems I
Fall 2022 Discussion 3B

1. Proofs

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is **linearly dependent** if there exists constants $c_1, c_2, \ldots\ c_n$ such that $\sum_{i=1}^{i=n} c_i \vec{v}_i = \vec{0}$ and at least one c_i is non-zero.

This condition intuitively states that it is possible to express any vector from the set in terms of the others.

(a) Suppose for some non-zero vector $\vec{x}$, $\mathbf{A}\vec{x} = \vec{0}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Answer:

Begin by defining column vectors $\vec{a}_1 \ldots \vec{a}_n$.

$$\mathbf{A} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

Thus, we can represent the multiplication $\mathbf{A}\vec{x}$ as

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \sum x_i \vec{a}_i = \vec{0}$$

Note that the equation above is the definition of linear dependence. That is, there exist coefficients, at least one which is non-zero, such that the sum of the vectors weighted by the coefficients is zero. These coefficients are the elements of the non-zero vector $\vec{x}$.

(b) For $\mathbf{A} \in \mathbb{R}^{m \times n}$, suppose there exist two unique vectors $\vec{x}_1$ and $\vec{x}_2$ that both satisfy $\mathbf{A}\vec{x} = \vec{b}$, that is, $\mathbf{A}\vec{x}_1 = \vec{b}$ and $\mathbf{A}\vec{x}_2 = \vec{b}$. Prove that the columns of $\mathbf{A}$ are linearly dependent.

Answer:

Let us consider the difference of the two equations:

$$\mathbf{A}\vec{x}_1 - \mathbf{A}\vec{x}_2 = \mathbf{A}(\vec{x}_1 - \vec{x}_2) = \vec{b} - \vec{b} = \vec{0}$$

Once again, we've reached the definition of linear dependence since $\vec{x}_1 - \vec{x}_2 \neq \vec{0}$. We can apply the results from part (a), setting $\vec{x} = \vec{x}_1 - \vec{x}_2$.

(c) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix for which there exists a non-zero $\vec{y} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{y} = \vec{0}$. Let $\vec{b} \in \mathbb{R}^m$ be some non zero vector. Show that if there is one solution to the system of equations $\mathbf{A}\vec{x} = \vec{b}$, then there are infinitely many solutions.

Answer: The key insight is to use the linearity of Matrix-vector multiplication.

By assumption, let $\vec{x}_1 \in \mathbb{R}^n$ be a solution to $\mathbf{A}\vec{x} = \vec{b}$. Then, for any $c \in \mathbb{R}$

$$\mathbf{A}(\vec{x}_1 + c\vec{y}) = \mathbf{A}\vec{x}_1 + \mathbf{A}(c\vec{y}) = \mathbf{A}\vec{x}_1 + c\mathbf{A}\vec{y} = \mathbf{A}\vec{x}_1 + \vec{0} = \mathbf{A}\vec{x}_1 = \vec{b}$$

where the first two equalities follow by linearity and the last two equalities follow from the assumptions that $\mathbf{A}\vec{y} = \vec{0}$ and that $\vec{x}_1$ is a solution to the system.

Hence, $\mathbf{A}(\vec{x}_1 + c\vec{y}) = \vec{b}$, implying that $(\vec{x}_1 + c\vec{y})$ is also a solution to $\mathbf{A}\vec{x} = \vec{b}$ for **any** constant c. Therefore, there are infinitely many solutions.

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices! Note that in this exercise we are applying a matrix transformation on each of the vertices of the unit square separately.

(a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by 15° and 30° respectively. Suggest some methods to rotate the unit square by 45° using only $\mathbf{T}_1$ and $\mathbf{T}_2$. How would you rotate the square by 60°? Your TA will show you the result in the iPython notebook.

Answer:

Apply $\mathbf{T}_1$ and $\mathbf{T}_2$ in succession to rotate the unit square by 45°. To rotate the square by 60°, you can either apply $\mathbf{T}_2$ twice, or if you prefer variety, apply $\mathbf{T}_1$ twice and $\mathbf{T}_2$ once.

(b) Find a single matrix $\mathbf{T}_3$ to rotate the unit square by 60°. Your TA will show you the result in the iPython notebook.

Answer: This matrix will look like the rotation matrix that rotates a vector by 60°. This matrix can be composed by multiplying $\mathbf{T}_1$ by $\mathbf{T}_1$ by $\mathbf{T}_2$ (or equivalently, $\mathbf{T}_2$ by $\mathbf{T}_2$).

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. To do this, consider rotating the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ by θ degrees using the matrix $\mathbf{R}$.

(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \ldots} = 1$.)

(*Hint: Use your trigonometric identities:* $\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a+b)$, $\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a+b)$.)

Answer:

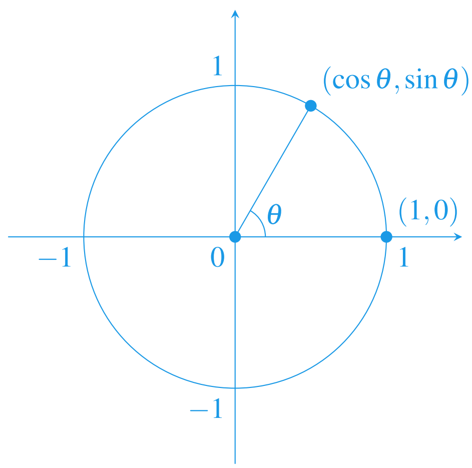
The reason the matrix is called a rotation matrix is because it transforms the unit vector $\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ to give $\begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}$.

Proof:

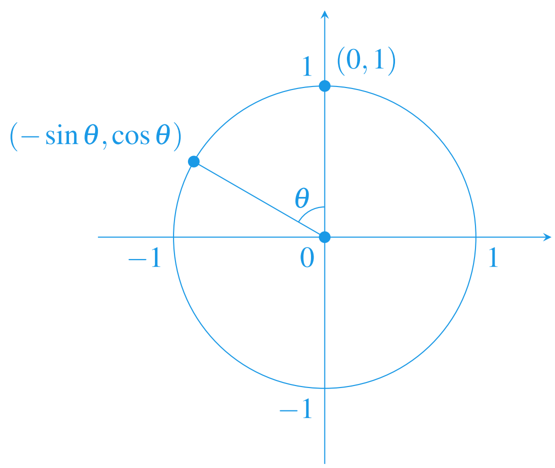
$$\begin{aligned}
\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} &= \cos\alpha \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \sin\alpha \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \cos\alpha\sin\theta + \sin\alpha\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}
\end{aligned}$$

Alternative solution:

Let's try to derive this matrix using trigonometry. Suppose we want to rotate the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by θ.



We can use basic trigonometric relationships to see that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotated by θ becomes $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. Similarly, rotating the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by θ becomes $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$:



We can also scale these pre-rotated vectors to any length we want, $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y \end{bmatrix}$, and we can observe graphically that they rotate to $\begin{bmatrix} x\cos\theta \\ x\sin\theta \end{bmatrix}$ and $\begin{bmatrix} -y\sin\theta \\ y\cos\theta \end{bmatrix}$, respectively. Rotating a vector solely in the x-direction produces a vector with both x and y components, and, likewise, rotating a vector solely in the y-direction produces a vector with both x and y components.

Finally, if we want to rotate an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$, we can combine what we derived above. Let x' and y' be the x and y components after rotation. x' has contributions from both x and y: $x' = x\cos\theta - y\sin\theta$. Similarly, y' has contributions from both components as well: $y' = x\sin\theta + y\cos\theta$. Expressing this in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, we've derived the 2-dimensional rotation matrix.

(d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition; we will visit inverses very soon in lecture!)

Answer:

Use a rotation matrix that rotates by $-60°$.

$$\begin{bmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{bmatrix}$$

(e) Use part (d) to obtain the rotation matrix that reverses the operation of a matrix that rotates a vector by θ. Multiply the reverse rotation matrix with the rotation matrix and vice-versa. What do you get?

Answer:

The reverse matrix is as follows:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

We can see that for any $\vec{v} \in \mathbb{R}^2$ that the product of the rotation matrix with $\vec{v}$ followed by the product of the reverse results in the original $\vec{v}$.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \vec{v} \right) = \vec{v}$$

(f) (For Practice) Next, we will look at reflection matrices. The matrix that reflects about the y axis is:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

What are the matrices that reflect a vector about the (i) x-axis, and (ii) line $x = y$?

Answer: The matrix that reflects about the x-axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the matrix that reflects about $x = y$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A natural question to ask is the following: does the *order* in which you apply these operations matter?

(g) Let's see what happens to a vector when we rotate it by $60°$ and then reflect it along the y-axis (matrix given in part (f)). Next, let's see what happens when we first reflect the vector along the y-axis and then rotate it by $60°$. You will need to multiply the corresponding rotation and reflection matrices in the correct order. Are the results the same?

Answer: The results are not the same. If you rotate some vector $\vec{v}$ and then reflect along the y-axis you get:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\cos(60°) & \sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

If you reflect along the y-axis and then rotate you get:

$$\begin{bmatrix} \cos(60^\circ) & -\sin(60^\circ) \\ \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\cos(60^\circ) & -\sin(60^\circ) \\ -\sin(60^\circ) & \cos(60^\circ) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

(h) Now, let's perform the operations in part (g) on the unit square in our iPython notebook. Are the results the same?

Answer: The results are not the same as shown in the iPython notebook.