EECS 16A Designing Information Devices and Systems I
Fall 2022 Discussion 3A

1. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

For each matrix multiplication problem, *if the product exists*, find the product by hand.
Otherwise, explain why the product does not exist.

(a) **A B** **Answer:** A is a 1×2 vector and B_1 is a 2×1 vector, so the product exists!
$\mathbf{AB} = 1 \cdot 3 + 4 \cdot 2 = 11$

(b) **C D** **Answer:** Since both **C** and **D** are 2×2 matrices, the product exists and is a 2×2 matrix.
$$\mathbf{CD} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 3 + 4 \cdot 2 & 1 \cdot 2 + 4 \cdot 1 \\ 2 \cdot 3 + 3 \cdot 2 & 2 \cdot 2 + 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}.$$

(c) **D C** **Answer:** Since both **C** and **D** are 2×2 matrices, the product exists and is a 2×2 matrix.
$$\mathbf{DC} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 2 \cdot 2 & 3 \cdot 4 + 2 \cdot 3 \\ 2 \cdot 1 + 1 \cdot 2 & 2 \cdot 4 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}.$$

(d) **C E** **Answer:** Since **C** is a 2×2 matrix and **E** is a 2×4 matrix, the product exists and is a 2×4 matrix.
$$\mathbf{CE} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 4 \cdot 4 & 1 \cdot 9 + 4 \cdot 3 & 1 \cdot 5 + 4 \cdot 2 & 1 \cdot 7 + 4 \cdot 2 \\ 2 \cdot 1 + 3 \cdot 4 & 2 \cdot 9 + 3 \cdot 3 & 2 \cdot 5 + 3 \cdot 2 & 2 \cdot 7 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}.$$

(e) **F E** (only note whether or not the product exists and optionally compute the product if it does) **Answer:**
Since **E** is a 2×4 matrix and **F** is a 4×3 matrix, the product does not exist.
This is because the number of columns in the first matrix (**F**) should match the number of rows in the second matrix (**E**) for this product to be defined.

(f) **E F** (only note whether or not the product exists and optionally compute the product if it does) **Answer:**

Since **E** is a 2×4 matrix and **F** is a 4×3 matrix, the product exists and is a 2×3 matrix.

$$\begin{aligned}
\mathbf{EF} &= \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot5+9\cdot6+5\cdot4+7\cdot3 & 1\cdot5+9\cdot1+5\cdot1+7\cdot2 & 1\cdot8+9\cdot2+5\cdot7+7\cdot2 \\ 4\cdot5+3\cdot6+2\cdot4+2\cdot3 & 4\cdot5+3\cdot1+2\cdot1+2\cdot2 & 4\cdot8+3\cdot2+2\cdot7+2\cdot2 \end{bmatrix} \\
&= \begin{bmatrix} 100 & 33 & 75 \\ 52 & 29 & 56 \end{bmatrix}
\end{aligned}$$

(g) **G H** (Practice on your own) **Answer:** Since **G** and **H** are both 3×3 matrices, the product exists and is another 3×3 matrix.

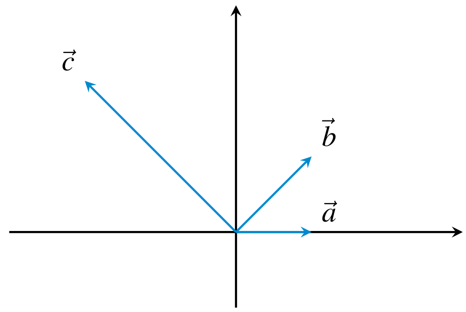
$$\mathbf{GH} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 8\cdot5+1\cdot1+6\cdot2 & 8\cdot3+1\cdot8+6\cdot3 & 8\cdot4+1\cdot2+6\cdot5 \\ 3\cdot5+5\cdot1+7\cdot2 & 3\cdot3+5\cdot8+7\cdot3 & 3\cdot4+5\cdot2+7\cdot5 \\ 4\cdot5+9\cdot1+2\cdot2 & 4\cdot3+9\cdot8+2\cdot3 & 4\cdot4+9\cdot2+2\cdot5 \end{bmatrix} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}.$$

(h) **H G** (Practice on your own) **Answer:** Since **H** and **G** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{GH} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} = \begin{bmatrix} 5\cdot8+3\cdot3+4\cdot4 & 5\cdot1+3\cdot5+4\cdot9 & 5\cdot6+3\cdot7+4\cdot2 \\ 1\cdot8+8\cdot3+2\cdot4 & 1\cdot1+8\cdot5+2\cdot9 & 1\cdot6+8\cdot7+2\cdot2 \\ 2\cdot8+3\cdot3+5\cdot4 & 2\cdot1+3\cdot5+5\cdot9 & 2\cdot6+3\cdot7+5\cdot2 \end{bmatrix} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}.$$

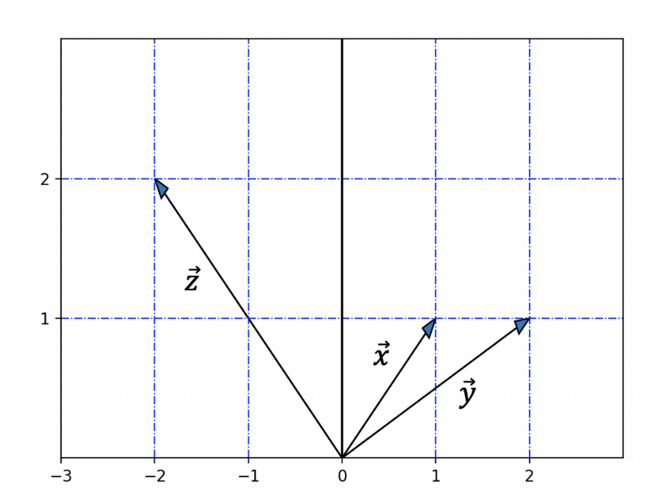
2. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y}=\begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z}=\begin{bmatrix}-2\\2\end{bmatrix}$. Draw these vectors on a sheet of paper.

Answer:



(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$, we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

Answer:

$$\alpha\vec{x}+\beta\vec{y}=\vec{z}$$

$$\alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$$

$$\begin{cases}\alpha+\beta\cdot 2=-2\\ \alpha+\beta=2\end{cases}$$

$$\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}-2\\2\end{bmatrix}$$

(c) Solve for α, β.

Answer: We start by writing the system in the augmented matrix form

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 1 & 1 & 2 \end{array}\right]$$

Then we solve the system using Gaussian Elimination. First, we subtract the second row by the first row:

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & -1 & 4 \end{array}\right]$$

Next, we multiply the second row by -1 to solve for β.

$$\left[\begin{array}{cc|c} 1 & 2 & -2 \\ 0 & 1 & -4 \end{array}\right]$$

We get $\beta = -4$. Then, we take the first row and subtract it by the second row*2.

$$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -4 \end{array}\right]$$

So the solution is $\alpha = 6$ and $\beta = -4$.

3. Span Basics

(a) What is span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer: span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ contains any vector $\vec{v}$ that can be written as

$$\vec{v} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

We realize that any vector whose last component is 0 can be written in this form and any vector whose last component is nonzero cannot. Hence, the required span is the set of all vectors that can be written in the form $\begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$.

(b) Is $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ in span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$?

Answer: From the definition of span, we know that if we can express $\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, it is in the span. Assume such a linear combination exists, and the two vectors are respectively multiplied by α_1 and α_2, we can set up such an equation

$$\alpha_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}5\\5\\0\end{bmatrix}$$

. We can solve this by writing it into augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

Solving this with Gaussian Elimination, we get

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 0 \end{array}\right]$$

Therefore, we know $\alpha_1 = \frac{5}{3}$, $\alpha_2 = \frac{5}{3}$, so we can conclude the statement is true.

(c) What is a possible choice for $\vec{v}$ that would make $\text{span}\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3$?

Answer: From part (a), we realize that any vector whose last component is 0 can be written as a linear combination of the two vectors already in the set. Hence, if we include, for example, $\begin{bmatrix}0\\0\\1\end{bmatrix}$ into the set, then we should be able to reach any vector in $\mathbb{R}^3$. Any vector whose last component is non-zero is a valid addition to the set to achieve the desired span.

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? ("Consistent" means there is at least one solution.)

$$\begin{bmatrix}1 & 2\\2 & 1\\0 & 0\end{bmatrix} \vec{x} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

Answer: For the system of linear equations to be consistent, there must exist some x such that the equality above holds. Performing matrix vector multiplication, we can rewrite the above equality as

$$x_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + x_2 \begin{bmatrix}2\\1\\0\end{bmatrix} = \vec{b}$$

The question now becomes: which vectors $\vec{b}$ can be written in the above form, i.e as a linear combination of the columns of A? This is exactly the definition of span, and the answer must be the same as that from part (a).