# EECS 16A Fa20: Compiled Discussion Checkoffs

December 2020

# 1 Week 1

## 1.1 Discussion 1A

**Question:**

If PG&E wants to determine the power consumption of five appliances, what's the minimum number of measurements they need to take?

**Answer:**

Minimum number of measurements for 5 appliances: 5

**Question:**

How did you set up your power measurements after the breaker was fixed?

**Answer:**

This depends on how you approached the problem, but there are 4 possible measurements. Any subset of three equations can help you determine the individual power consumption of each device. The four possible measurements are:

$$\begin{aligned} m_1 &= x_F \\ m_2 &= x_F + x_{TV} \\ m_3 &= x_F + x_{AC} \\ m_4 &= x_F + x_{TV} + x_{AC} \end{aligned}$$

## 1.2 Discussion 1B

**Question:**

What is the notation for the set containing this vector: $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$?

(A) $\mathbb{V}^2$

(B) $\mathbb{R}^3$

(C) $\mathbb{R}^4$

(D) $\mathbb{V}^3$

**Answer:**

(B). The $\mathbb{R}$ indicates that the entries are real numbers, while the superscript indicates how many entries there are in the vector.

**Question:**

Suppose you have a system of linear equations with 2 equations and 2 unknowns. How could you graphically interpret this system if there is no solution?

**Answer:**

Plotting each linear equation, there are two lines. Solutions are the set of intersections. With no solutions, there is no intersection so the lines must be parallel.

# 2 Week 2

## 2.1 Discussion 2B

**Question:**

If you have two vectors $\vec{v}_1$ and $\vec{v}_2$ that are not multiples of each other that lie in $\mathbb{R}^2$ space, what is the span of the two vectors $\{\vec{v}_1, \vec{v}_2\}$?

**Answer:**

$\mathbb{R}^2$ - The $\mathbb{R}$ indicates that the entries are real numbers. Two linearly independent vectors that lie in $\mathbb{R}^2$ can take up all the space in $\mathbb{R}^2$ by some linear combinations of them.

**Question:**

(True/False) A system of equations with more unknowns than equations will never have a unique solution. Explain.

**Answer:**

True. When performing Gaussian Elimination, you won't be able to form a pivot in every column (with a fat/wide matrix - # of rows < # of columns). The most pivots you can have equals to number of rows. Therefore, you can't find a unique solution when there are more unknowns than equations.

# 3 Week 3

## 3.1 Discussion 3A

**Question:**

Are matrices commutative in general? How about rotation matrices (in 2D)?

**Answer:**

No, general matrices cannot commute. But yes, rotation matrices do commute (in 2D). (Rotation matrices in higher dimensions do not commute in general - try rotating your hand along two different axes in space - you might notice your hand ends up in two different positions)

## 3.2 Discussion 3B

**Question:**

What conditions are required for the matrix product $\mathbf{AB}$ to be well-defined (possible) for matrices $\mathbf{A}$ and $\mathbf{B}$? How about for $\mathbf{BA}$ to be defined?

**Answer:**

For $\mathbf{AB}$ it is necessary that the horizontal dimension of $\mathbf{A}$ (# of columns) matches the vertical dimension of $\mathbf{B}$ (# of rows). If $\mathbf{A}$ has dimensions $m \times n$ (# rows $\times$ # columns) and $\mathbf{B}$ has $p \times q$, then $\mathbf{AB}$ requires $n = p$ and $\mathbf{BA}$ requires $q = m$. (Extra tip - you can see if this makes sense by trying to compute an entry, which takes the inner product of the rows of the left matrix and the columns of the right matrix!)

**Question:**

How do you compute the state vector after $n$ time-steps under a transition matrix?

**Answer:**

You compute the product $\underbrace{\mathbf{A}\mathbf{A}\cdots\mathbf{A}}_{n \text{ times}}\vec{x} = \mathbf{A}^n\vec{x}$, where $n$ is the number of time steps, $\vec{x}$ is the state vector and $\mathbf{A}$ represents the transition matrix.

# 4 Week 4

## 4.1 Discussion 4A

**Question:**

Does the inverse of a square matrix always exist? Why or why not?

**Answer:**

No, not all square matrices have an inverse. The columns of square matrices need to be linearly independent for its inverse to exist.

### 4.2 Discussion 4B

**Question:**

Does $\vec{0}$ always exist in every vector subspace?

**Answer:**

Yes.$\vec{0}$ exists in all vector subspaces. (Caveat! $\vec{0}$ from $\mathbb{R}^m$ is not the same as $\vec{0}$ from $\mathbb{R}^n$, $n \neq m$ - they will have a different number of zeroes! Make sure to disambiguate scalar zeroes and vector zeroes. Similar idea of the identity matrix - we typically just use the same symbol $\mathbf{I}$ and can mean different sized matrices. Just be sure to track what size it has.)

**Question:**

Let $\mathcal{V}$ be a vector space with dimension $n$. Let $\mathbb{S}$ be a set of vectors that is a subset of $\mathcal{V}$. $\mathbb{S}$ has $n-1$ linearly independent vectors. Which of the following is TRUE? (Check all that apply)

(A) $\dim(\text{span}(\mathbb{S})) < \dim(\mathcal{V})$

(B) $\dim(\text{span}(\mathbb{S})) = \dim(\mathcal{V})$

(C) $\mathbb{S}$ is a basis of $\mathcal{V}$

(D) $\mathbb{S}$ is not a basis of $\mathcal{V}$

**Answer:**

(A) and (D) are true. The dimension of $\text{span}(\mathbb{S})$ is $n-1$, whereas the dimension of $\mathcal{V}$ is $n$; therefore dimension of $\text{span}(\mathbb{S})$ is less than the dimension of $\mathcal{V}$ and $\mathbb{S}$ is not a basis of $\mathcal{V}$. (Fails the condition that $\mathbb{S}$ must generate $\mathcal{V}$ when we take the span of $\mathbb{S}$.)

## 5 Week 5

### 5.1 Discussion 5A

**Question:**

Write out the formula for the determinant of a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

**Answer:**

The formula is $\det(A) = ad - bc$ (Mnemonic - for $2\times2$ matrices - product of diagonal minus product of off-diagonal.)

### 5.2 Discussion 5B

**Question:**

What are the eigenvalues for the $2 \times 2$ rotation matrix by angle $\theta$? (For reference $e^{\pm ix} = \cos(x) \pm i\sin(x)$, where $i = \sqrt{-1}$.)

**Answer:**

$\lambda = e^{i\theta}$ AND $e^{-i\theta}$. Only when $\theta = 0°$ or $\theta = 180°$ are they real eigenvalues, which correspond to either the identity matrix or the inversion matrix (all we mean by this is $-\mathbf{I}$, which is equivalent to multiplying by $-1$)!

Extra: the calculation follows with the comment that it is not important to remember the eigenvalues of the rotation matrix specifically, but more so how you compute eigenvalues in general. To find the eigenvalues, we need to solve for the $\lambda$ that make $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

$$\det\left(\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = 0$$
$$\det\left(\begin{bmatrix}\cos\theta-\lambda & -\sin\theta\\ \sin\theta & \cos\theta-\lambda\end{bmatrix}\right) = 0$$
$$(\cos\theta-\lambda)^2 + \sin^2\theta = 0$$
$$(\lambda-\cos\theta)^2 = -\sin^2\theta$$
$$\lambda-\cos\theta = \pm i\sin\theta$$
$$\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

**Question:**

If matrix $\mathbf{M}$ has an eigenvalue of $\lambda = 0$, what can you conclude about its null space?

**Answer:**

Matrix $\mathbf{M}$ has a null space that is not just the set $\{\vec{0}\}$ (alternatively, a nontrivial null space), since for eigenvalue $\lambda = 0$ we know there is a nonzero vector $\vec{x}$ for which $\mathbf{M}\vec{x} = \lambda\vec{x} = 0\vec{x} = \vec{0}$. Since $\mathbf{M}\vec{x} = \vec{0}$, $\vec{x}$ is in the null space of $\mathbf{M}$.

# 6 Week 6

## 6.1 Discussion 6A

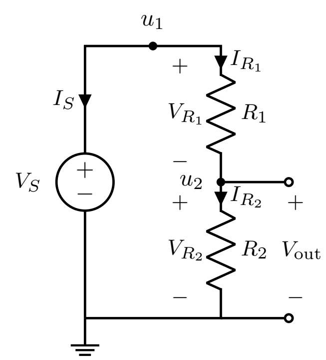
**Question:**

If an $n \times n$ matrix, $\mathbf{A}$, has a non-trivial nullspace, then $\mathbf{A}$ is not invertible.

**Answer:**

True - the proof is provided in the worksheet solution.

## 6.2 Discussion 6B




**Question:**

What is the output voltage of a voltage divider circuit?

(A) $V_{\text{out}} = \frac{R_1}{R_1+R_2}V_S$

(B) $V_{\text{out}} = \frac{R_2}{R_1+R_2}V_S$

(C) $V_{\text{out}} = V_S$

(D) $V_{\text{out}} = 0$

**Answer:**

(B). $V_{\text{out}} = \frac{R_2}{R_1+R_2}V_S$. For a detailed solution refer to the discussion worksheet.

**Question:**

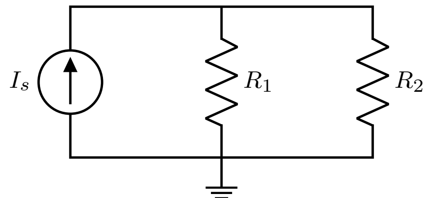
What is the range of values (upper and lower limit) the output voltage, $V_{\text{out}}$, the above circuit can have?

**Answer:**

$0 < V_{\text{out}} < V_S$. As shown by the formula in the above check, this circuit can only generate voltages that are below $V_S$ and above 0 (with $R_1$, $R_2$, $V_S$ all positive). Hence it has the name voltage divider, since it "divides" voltage down from the value of the voltage source $V_S$ (Alternatively - you can think of it as splitting the voltage source over the two resistors - the sum of $V_{R_1}$ and $V_{R_2} = V_{\text{out}}$ will be $V_S$). An interesting property to notice is that if $R_2 \gg R_1$ then $V_{\text{out}} \to V_S$ and if $R_2 \ll R_1$ then $V_{\text{out}} \to 0$.

# 7 Week 7

## 7.1 Discussion 7A



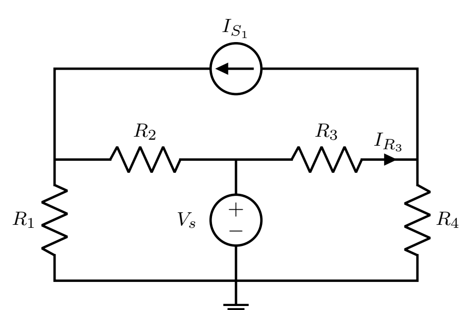

**Question:**

For a current divider with resistors $R_1 = 560\Omega$, $R_2 = 120\Omega$ , and $I_s = 4.5\text{mA}$, what are the currents through $R_1$, $I_1$, and the current through $R_2$, $I_2$? (assume you've labeled the circuits so that the currents are positive - values are given with 2 significant figures)

**Answer:**

$$I_1 = \frac{R_2}{R_1 + R_2} I_S = \frac{120}{680} 4.5\text{mA} = 0.79\text{mA}$$
$$I_2 = \frac{R_1}{R_1 + R_2} I_S = \frac{560}{680} 4.5\text{mA} = 3.7\text{mA}$$

Notice that the current through the larger resistor ($I_1$ through $560\Omega$) is smaller, and the current through the smaller resistor ($I_2$ through $120\Omega$) is larger.




**Question:**

Which of the four nodes in this circuit have unknown node voltages you would solve for by NVA?

(A) Left node connected to $R_1$, $R_2$, $I_{S_1}$

(B) Right node connected to $R_3$, $R_4$, $I_{S_1}$

(C) Middle node connected to $V_S$, $R_2$, $R_3$

(D) Ground node connected to $R_1$, $V_S$, $R_4$

**Answer:**

(A), (B). The left and right nodes have unknown node voltages. The ground node is set to 0V because it is the reference or ground. The middle node is set to a node voltage of $V_S$ because the voltage source keeps the (node) voltage (difference) between the ground node and the middle node to be $V_S$. $V_S = u_{\text{middle}} - u_{\text{bottom}} = u_{\text{middle}} - 0\text{V} \implies u_{\text{middle}} = V_S$.

## 7.2 Discussion 7B

**Question:**

If you follow passive sign convention, what is the sign of the power calculated for an element that is dissipating power?

**Answer:**

If following passive sign convention, a circuit element will have a positive power value calculated when it is dissipating power. This means that energy is exiting the circuit through that element as heat, or another form of energy (resistors dissipate energy as heat, and a motor can dissipate electrical energy into the mechanical energy of motion). A circuit element will have a negative power calculated when it is supplying power to the circuit. This means that electrical energy is entering the circuit through that component and being distributed to other components.

**Question:**

Which of these does an ideal voltmeter behave like?

(A) Resistor, $0 < R < \infty$

(B) Wire

(C) Open

(D) Voltage source

(E) Current source

**Answer:**

(C). See answer to below.

**Question:**

Which of these does an ideal ammeter behave like?

(A) Resistor, $0 < R < \infty$

(B) Wire

(C) Open

(D) Voltage source

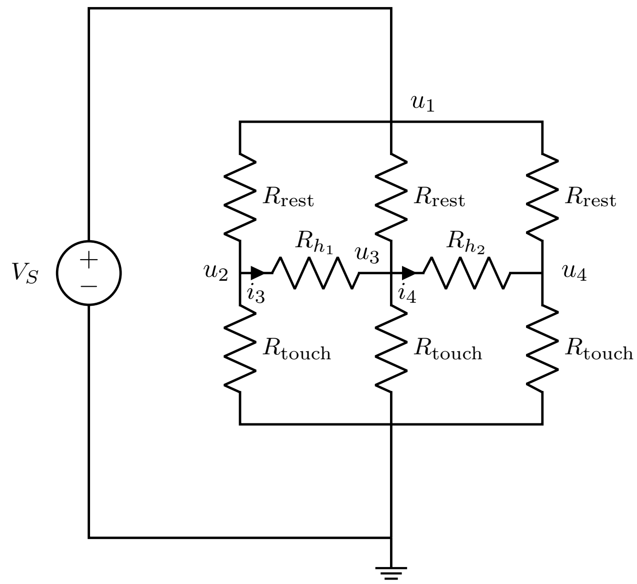
(E) Current source

**Answer:**

(B). An ideal voltmeter will behave like an open (infinite resistance), and an ideal ammeter will behave like a short/a wire (zero resistance). If a meter behaves as if it has some finite non-zero resistance, energy can be dissipated on it and it can affect the behavior of the circuit in an undesirable way. One must be conscious of how measurements may influence circuit behavior.

# 8 Week 8

## 8.1 Discussion 8A




**Question:**

What is the relationship between node $u_2$, $u_3$, and $u_4$?

(A) $u_2 = u_3 = u_4$

(B) $u_2 < u_3 < u_4$

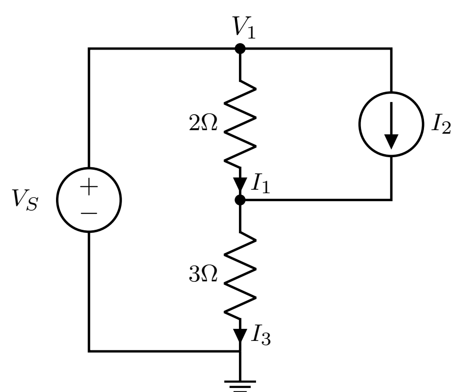
(C) $u_2 > u_3 > u_4$

(D) Can't tell unless given the relationship between $R_{h1}$ and $R_{h2}$.

**Answer:**

$u_2 = u_3 = u_4$. This is the circuit similar to the "interesting circuit" in Note 14.

## 8.2 Discussion 8B

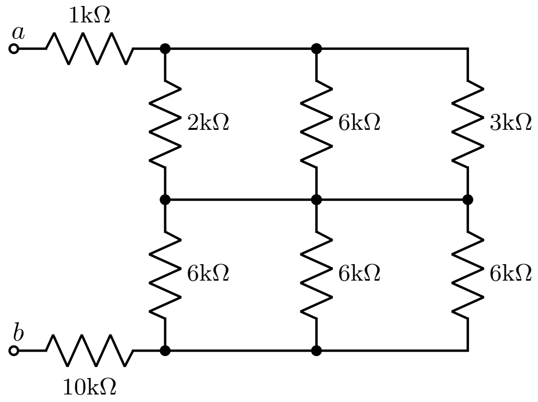



**Question:**

Let $V_s = 5\text{V}$, $I_2 = 1\text{A}$, what is the node voltage at $V_2$ when only $V_s$ is turned on ($I_2$ is off)?

**Answer:**

$V_2 = 3\text{V}$ when only $V_s$ is turned on. $I_2$ acts like an open circuit when it's turned off, so you can solve $V_2$ by applying voltage divider: $V_2 = 5\text{V} \cdot \frac{3}{2+3}$.




**Question:**

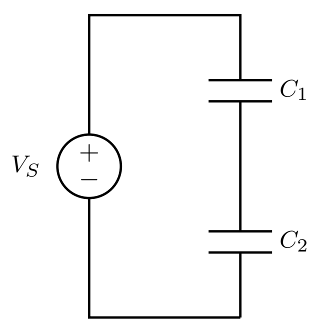
What is the equivalent resistance between points $a$ and $b$?

**Answer:**

The equivalent resistance is $1\text{k}\Omega + (2\text{k}\Omega \parallel 6\text{k}\Omega \parallel 3\text{k}\Omega) + (6\text{k}\Omega \parallel 6\text{k}\Omega \parallel 6\text{k}\Omega) + 10\text{k}\Omega$.

# 9 Week 9

## 9.1 Discussion 9A




**Question:**

For capacitors $C_1$ and $C_2$ wired in series with a source voltage $V_s$ charging both capacitors, how are the charges on each capacitor distributed? (circuit provided below) (Assume that the capacitors are initially uncharged)

(A) There are equal charges on each capacitor ($+Q$ on all top plates, $-Q$ on all bottom plates)

(B) The charges are divided based on the capacitance values $C_1$ and $C_2$.

(C) All of the charge is on Capacitor 1 ($+Q$ on top plate, $-Q$ on bottom, no charge on either of the $C_2$ plates)

(D) All of the charge is on Capacitor 2 ($+Q$ on top plate, $-Q$ on bottom, no charge on either of the $C_1$ plates)

**Answer:**

(A). There are equal charges on each capacitor ($+Q$ on all top plates, $-Q$ on all bottom plates). This is required since the wire segment is assumed to start charge neutral. Since this segment is isolated, it must remain charge neutral, thus each capacitor must always have matching charges.

**Question:**

Based on the circuit below, how do you expect the voltage across the capacitor to change in time? (assume no initial charge on $C$ at $t = 0$ and that $I_S$ is constant).

(A) linear: $V(t) = a \cdot t$

(B) quadratic: $V(t) = a \cdot t^2$

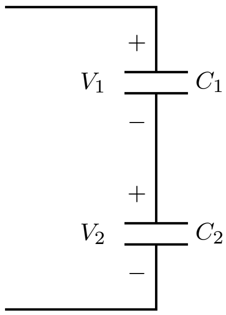
(C) exponential: $V(t) = e^{at}$

(D) constant: $V(t) = V_0$

**Answer:**

(A). Linear, specifically $a = \frac{I_s}{C}$. We can see this from the capacitor equation: $Q = CV \implies I = C\frac{dV}{dt} \implies$ the voltage must have a constant slope.

### 9.2 Discussion 9B




**Question:**

At what location (which node) are we guaranteed that charge is conserved? Assume that the top and bottom nodes are connected to some other circuit or circuit element.

**Answer:**

Only the middle node is guaranteed to have charge conservation apply - it is a called a floating node, which is a node from which charge cannot leave or enter. The top and bottom nodes may have charge leave or enter depending on what the capacitor is hooked up to.

**Question:**

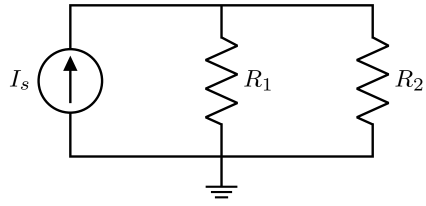
If two capacitors $C_1$, $C_2$ start disconnected (with $C_1$ charged $+3\mu C$) then are linked in parallel, what do you know about the circuit?

**Answer:**

Both capacitors have matching element voltages. This is the definition of a parallel system, whereas the charges will distribute based on the capacitance values.

## 10 Week 10

### 10.1 Discussion 10A



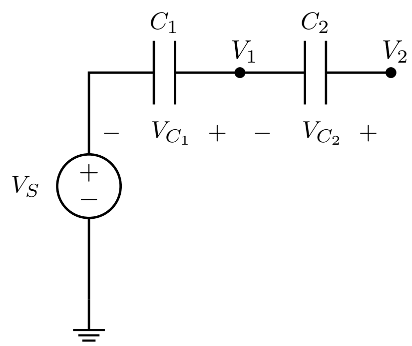

**Question:**

In the circuit shown above how much of the current goes through resistor $R_1$ and how much through resistor $R_2$?

**Answer:**

The current divider formula gives: $I_{R_1} = I_s \cdot \frac{R_2}{R_1+R_2}$, $I_{R_2} = I_s \cdot \frac{R_1}{R_1+R_2}$, Mnemonic rule: when trying to find $I_{R_1}$ use $R_2$ in the numerator (the opposite from the voltage divider case!) Intuition: If $R_1 \gg R_2 \implies I_{R_2} = I_s \cdot \frac{R_1}{R_1+R_2} \approx I_s$. Almost all of the current flows through $R_2$ instead of $R_1$ (takes the path of least resistance), use the other resistor value in the numerator.

Can prove formally by: using the equivalent resistance formula to calculate the voltage across the resistors; $V_{R_1} = V_{R_2} = I_s \cdot R_{\text{eq}} = I_s \cdot \frac{R_1 \cdot R_2}{R_1 + R_2}$, then $I_{R_1} = \frac{V_{R_1}}{R_1}$, $I_{R_2} = \frac{V_{R_2}}{R_2}$.




**Question:**

In the circuit above which nodes are floating? (A floating node, is a node that is only connected to capacitor plates or op-amp or comparator inputs).

**Answer:**

Following the definition, we can see that both nodes $V_1$ and $V_2$ are floating. The other two nodes are connected to the voltage source, so charge can flow in and out of them through the source.

## 10.2 Discussion 10B

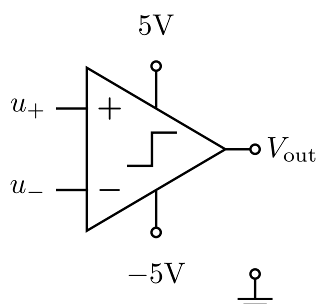
**Question:**

List the 2 golden rules of ideal op amps in negative feedback.

**Answer:**

1. If the op amp is in negative feedback, $u_+ = u-$
2. $i_+ = i_- = 0$

In words: If in negative feedback, the node voltages at the "+" and "-" terminals are equal (even though they are most often not physically connected). No current will flow into either of the op amp input terminals.




**Question:**

If in the above circuit we have $u_+ > u_-$ what is the output voltage, $V_{\text{out}}$?

**Answer:**

$V_{\text{out}} = 5V$. In a comparator when the voltage at the positive terminal is greater than the input at the negative terminal, the output rails to the positive supply, $V_{\text{DD}}$, which in this case is $5V$.

# 11 Week 11

## 11.1 Discussion 11A

**Question:**

Why is a voltage buffer needed between two voltage dividers? What happens without?

**Answer:**

A voltage buffer allows us to choose (i.e. design) the resistor values for two voltage dividers separately and chain them. A voltage divider connected to another voltage divider can affect the behavior of the divider it is connected to, by changing effective (more precisely, equivalent) resistances, and thereby changing the resistor ratios that were separately designed and therefore the output voltages.

**Question:**

T/F - Non-inverting amplifiers alone can achieve all positive scalings of an input voltage - we can multiply our input voltage by any positive number.

**Answer:**

False: The formula indicates that the voltage gain (voltage multiplier from input voltage to output voltage) is at least 1: $1 + \frac{R_2}{R_1} > 0$. So we can only achieve output voltages that are related to input voltages by a gain or multiplier of value greater than 1. For example, we cannot multiply by 1/2 with only a non-inverting amplifier.

**Question:**

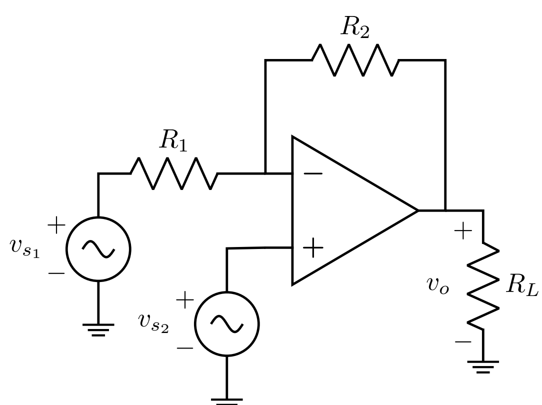
T/F - You can use a voltage summer alone to achieve a voltage $V_{\text{out}} = 3V_1 + 2V_2$.

**Answer:**

False: The formula has weights of a resistor over the sum of resistors. These factors are always less than one, and the two weights on each voltage sum to 1. So we cannot achieve voltage sums with weights higher than one unless we scale the voltages with another module (e.g. an op amp circuit) that takes the result of the voltage summer and multiplies it by a constant.

# 12 Week 12

## 12.1 Discussion 12A




**Question:**

Identify what the op-amp circuit is when zero-ing out $V_{s2}$.

(A) Inverting Amplifier

(B) None of the above

(C) Unity Gain Buffer

(D) None of the above

**Answer:**

(A). When zero-ing out $V_{s2}$, the op-amp circuit is an inverting amplifier.

**Question:**

What is the inner product for the following two vectors?

$$\begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$$

**Answer:**

The inner product is $(-2) \cdot 3 + 1 \cdot (-5) + 6 \cdot 2 = 1$.

### 12.2 Discussion 12B

**Question:**

Consider two vectors: $\vec{x}$ and $\vec{y}$. Does $\text{corr}_{\vec{x}}(\vec{y})$ equal to $\text{corr}_{\vec{y}}(\vec{x})$? In other words, is correlation commutative?

**Answer:**

No, correlation is not commutative.

**Question:**

How many spheres do you need to have intersect to uniquely define a point in 3D?

**Answer:**

You need 4 spheres to uniquely define a point in 3D. From the "finding Mr. Muffin" discussion problem, you know that 3 circles can uniquely identify a point in 2D. This extends to 4 spheres to 3D, 5 hyper-spheres to 4D, ... and so on.

## 13 Week 13

### 13.1 Discussion 13A

**Question:**

Does a projection of a vector, $\vec{b}$, onto a vector, $\vec{a}$, depend on the length of the vector $\vec{a}$?

**Answer:**

No, only the direction matters - If you scale the vector $\vec{a}$, as long as it points along the same line, we should expect to see the same projection vector.

**Question:**

What happens if you project a vector onto a vector along the same direction?

(A) The projected vector is the same as the original vector

(B) The projected vector is longer in length than the original vector

(C) The projected vector is shorter in length than the original vector

**Answer:**

(A). The vector projection will be the same as the original vector, both in direction and length. Projections cannot lengthen a vector. (You can use Cauchy-Schwarz to show this - compute the length of a projection of a vector onto another vector)

**Question:**

What is the difference between the $\vec{\hat{x}}$ vector, and the $A\vec{\hat{x}}$ vector in least squares?

**Answer:**

The $\vec{\hat{x}}$ vector is the vector with entries that tell us how to scale and add the columns of the $A$ matrix to get the best approximation of a vector, $\vec{b}$. $\mathbf{A}\vec{\hat{x}}$ is the best approximation of the vector $\vec{b}$.

# 14 Week 14

## 14.1 Discussion 14A

**Question:**

What is a condition that needs to hold so that we can apply the least squares approximation algorithm?

**Answer:**

$\mathbf{A}^\top\mathbf{A}$ needs to be invertible. As a condition on $\mathbf{A}$, $\mathbf{A}$ must have linearly independent columns. This makes least squares not have a unique solution when $\mathbf{A}$ is wide (wideness guarantees linear dependence of columns).

**Question:**

What is the minimum number of "noisy" $(x, y)$ pairs that we need to fit the following polynomial using least squares?

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

**Answer:**

As long as we have at least 5 data points of $(x, y)$ such that the $A$ matrix has linearly independent columns (this will lead to a square matrix), then we will find that we can perform least squares - note we could also perform Gaussian elimination, and the linear independence guarantees us a unique solution. Note that the least squares solution for $\mathbf{A}$, a $5 \times 5$ matrix with linearly independent columns, will be the same as that of gaussian elimination: $\hat{\vec{x}} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\vec{b} = \mathbf{A}^{-1}(\mathbf{A}^\top)^{-1}\mathbf{A}^\top\vec{b} = \mathbf{A}^{-1}\mathbf{I}\vec{b} = \mathbf{A}^{-1}\vec{b}$.

(The property used above is that inverses of a product of two invertible matrices will be the product of the inverses individually in the reverse order - verify by checking that such two expressions would multiply out to identity - $\mathbf{AB} \cdot \mathbf{B}^{-1}\mathbf{A}^{-1}$.)

## 14.2 Discussion 14B

**Question:**

Consider two vectors: $\vec{x}$ and $\vec{y}$. Does $\text{corr}_{\vec{x}}(\vec{y})$ equal to $\text{corr}_{\vec{y}}(\vec{x})$? In other words, is correlation commutative?

**Answer:**

Correlation is not commutative.

**Question:**

Are the following statements equivalent for a tall matrix $\mathbf{A}$ ($m \times n$, $n$ cols, $m$ rows with $n < m$)?

1. $\mathbf{A}^\top\mathbf{A}$ is invertible
2. $\mathbf{A}$ has linearly independent columns

**Answer:**

Yes. It was proven in lecture that $\mathcal{N}(\mathbf{A}^\top\mathbf{A}) = \mathcal{N}(\mathbf{A})$. Forward direction: If $\mathbf{A}^\top\mathbf{A}$ is invertible, then $\mathbf{A}^\top\mathbf{A}$ has linearly independent columns (The only solution to $\mathbf{A}^\top\mathbf{A}\vec{x} = \vec{0}$ will be $\vec{x} = \vec{0}$ since we can multiply on both sides by the inverse. Since this is the trivial nullspace for $\mathbf{A}^\top\mathbf{A}$, then $\mathcal{N}(\mathbf{A}) = \{\vec{0}\}$ as well. If $\mathbf{A}$ has a trivial nullspace, it will also have linearly independent columns.

Backward direction: If $\mathbf{A}$ has linearly independent columns, then $\mathcal{N}(\mathbf{A}) = \{\vec{0}\} = \mathcal{N}(\mathbf{A}^\top\mathbf{A})$. If $\mathbf{A}^\top\mathbf{A}$ has a trivial nullspace, it has linearly independent columns. If the square matrix $\mathbf{A}^\top\mathbf{A}$ has linearly independent columns, it is invertible. Thus: $\mathbf{A}^\top\mathbf{A}$ is invertible if and only if the columns of $\mathbf{A}$ are linearly independent.