EECS 16A Designing Information Devices and Systems I
Fall 2020 Homework 3

This homework is due September 18, 2020, at 23:59.
Self-grades are due September 21, 2020, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw3.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit each file to its respective assignment on Gradescope.

The second to last question on this homework gives you a chance to practice the zoom proctoring setup for exams. You may want to start with this question

1. **Reading Assignment**

   For this homework, please read Notes 3 and 4. These notes will give you an overview of linear independence, span, and an introduction to thinking about and writing proofs. You are always welcome and encouraged to read beyond this as well. Write a paragraph about how you can use the strategies in the notes to tackle proof questions.

2. **Kinematic Model for a Simple Car**

   Learning Objective: *Many real world systems are not actually linear and have more complex behaviors. However, linear models can approximate non-linear systems under certain conditions.*

   Building a self-driving car first requires understanding the basic motions of a car. In this problem, we will explore how to model the motion of a car.

   There are several models that we can use to model the motion of a car. Assume we use a kinematic model, described in the following four equations and Figure 1.

$$x[k+1] = x[k] + v[k]\cos(\theta[k])\Delta t \tag{1}$$
$$y[k+1] = y[k] + v[k]\sin(\theta[k])\Delta t \tag{2}$$
$$\theta[k+1] = \theta[k] + \frac{v[k]}{L}\tan(\phi[k])\Delta t \tag{3}$$
$$v[k+1] = v[k] + a[k]\Delta t \tag{4}$$

   where

   - k, a nonnegative integer, indicates the time step at which we measure each variable (e.g. $v[k]$ is the speed at time step k and $v[k+1]$ is the speed at the following time step)

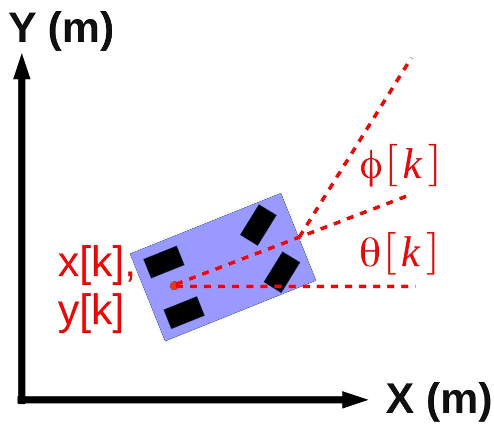


Figure 1: Vehicle Kinematic Model

- $x[k]$ and $y[k]$ denote the coordinates of the vehicle (meters)
- $\theta[k]$ denotes the heading of the vehicle, or the angle with respect to the x-axis (radians)
- $v[k]$ is the speed of the car (meters per second)
- $a[k]$ is the acceleration of the car (meters per second squared)
- $\phi[k]$ is the steering angle input we command (radians)
- Δt is a constant measuring the time difference (in seconds) between time steps $k+1$ and k
- L is a constant and is the length of the car (in meters)

For this problem, let L be 1.0 meter and Δt be 0.1 seconds.

The variables $x[k], y[k], \theta[k], v[k]$ describe the **state** of the car at time step k. The state captures all the information needed to fully determine the current position, speed, and heading of the car. The **inputs** at time step k are $a[k]$ and $\phi[k]$. These are provided by the driver. The current value of these inputs, along with the current state of the vehicle, will determine the state of the vehicle at the next time step.

We note that the problem is nonlinear, due to the sine, cosine and tangent functions, as well as terms including the product of states and inputs.

The purpose of this problem is to show that we can approximate a nonlinear model with a simple linear model and do reasonably well. This is why, despite many systems being nonlinear, linear algebra tools are widely used in practice.

For Parts (b) - (d), fill out the corresponding sections in prob3.ipynb.

(a) We assume that the car has a small heading, θ, which is a **very small but nonzero** value, and that the steering angle ϕ is also **very small but nonzero**. In this case, we could use the following approximations:

$$\begin{aligned}\sin(\alpha) &\approx 0,\\ \cos(\alpha) &\approx 1,\\ \tan(\alpha) &\approx 0.\end{aligned}$$

where α is the small angle of interest, and $\approx$ means "approximately equal to". Here, we use a very simple approximation for small angles; in later classes, you may learn better approximations.

Draw, by hand, graphs of of $\sin(\alpha)$ and $\cos(\alpha)$, for α ranging from $-\pi$ to π. Using these graphs can you justify the approximation we are making for small values of α?

Solution:

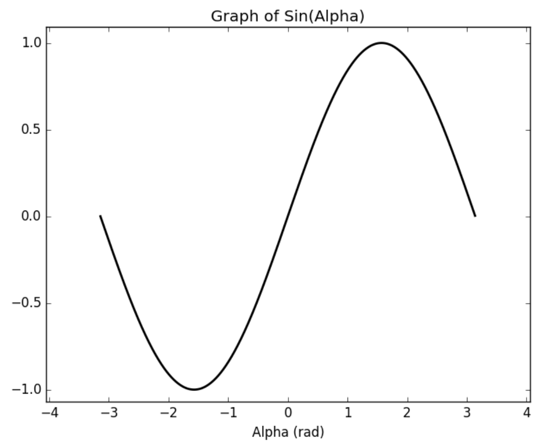


Figure 2: Graph of $sin(\alpha)$

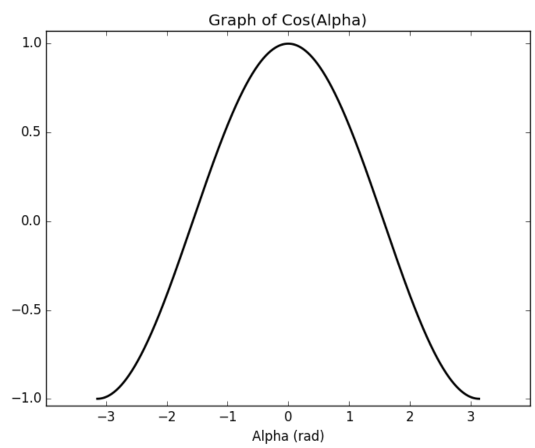


Figure 3: Graph of $cos(\alpha)$

From Fig. 2, we see that sine is close to zero for angles close to zero. From Fig. 3, we see that cosine is close to one for angles close to zero,

(b) Applying the approximation described in the previous part, write down a system of linear equations that approximates the nonlinear vehicle model given above in Equations (1) to (4). In particular, find the 4 x 4 matrix **A** and 4 x 2 matrix **B** that satisfy the equation given below.

$$\begin{bmatrix} x[k+1] \\ y[k+1] \\ \theta[k+1] \\ v[k+1] \end{bmatrix} = \mathbf{A} \begin{bmatrix} x[k] \\ y[k] \\ \theta[k] \\ v[k] \end{bmatrix} + \mathbf{B} \begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix}$$

Hint: *Start with simplifying Equations* (1) *to* (4)*, using the* $\sin(\alpha)$, $\cos(\alpha)$*, and* $\tan(\alpha)$ *approximations from part (a) only to approximate nonlinear terms. Do NOT replace* ϕ*,* θ *with 0 unless it is for the* $\sin(\alpha)$, $\cos(\alpha)$*, and* $\tan(\alpha)$ *approximations.*

Solution: In the region where both θ and ϕ are very small, we get that:

$$\begin{aligned}\sin(\theta) &\approx 0,\\ \cos(\theta) &\approx 1,\\ \tan(\phi) &\approx 0.\end{aligned}$$

So the vehicle equations simplify to a linear form:

$$\begin{aligned}x[k+1] &= x[k] + v[k]\Delta t\\ y[k+1] &= y[k] + 0\\ \theta[k+1] &= \theta[k] + 0\\ v[k+1] &= v[k] + a[k]\Delta t\end{aligned}$$

This corresponds to the following linear model:

$$\begin{bmatrix} x[k+1] \\ y[k+1] \\ \theta[k+1] \\ v[k+1] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \Delta t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x[k] \\ y[k] \\ \theta[k] \\ v[k] \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \end{bmatrix} \begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix}$$

(c) Suppose we drive the car so that the direction of travel is aligned with the x-axis, and we are driving nearly straight, i.e. the steering angle is $\phi[k] = 0.0001$ radians. (Driving exactly straight would have the steering angle $\phi[k] = 0$ radians.) The initial state and input are:

$$\begin{bmatrix} x[0] \\ y[0] \\ \theta[0] \\ v[0] \end{bmatrix} = \begin{bmatrix} 5.0 \\ 10.0 \\ 0.0 \\ 2.0 \end{bmatrix}$$

$$\begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0001 \end{bmatrix}$$

You can use these values in the IPython notebook to compare how the nonlinear system evolves in comparison to the linear approximation that you made. The IPython notebook simulates the car for ten time steps. Are the trajectories similar or very different? Why?

Solution: Yes, the linear model is a good approximation. This is since we are looking at a state and input that satisfy the two approximations we made with θ and ϕ. For example, the nonlinear model predicts that the state at time 10 is:

$$\begin{bmatrix} x[10] \\ y[10] \\ \theta[10] \\ v[10] \end{bmatrix} = \begin{bmatrix} 7.449 \\ 10.00 \\ 0.0000245 \\ 3 \end{bmatrix}$$

The state after 10 steps predicted by the linear model is very close.

$$\begin{bmatrix} x[10] \\ y[10] \\ \theta[10] \\ v[10] \end{bmatrix} = \begin{bmatrix} 7.45 \\ 10.0 \\ 0.0 \\ 3 \end{bmatrix}$$

From the plot, we see that both models show the vehicle moving along the x-axis and agree relatively well.

The linear model captures the state evolution well in the case where we drive mostly straight with a tiny heading angle. The models are close because the heading angle is tiny enough that the approximations are valid.

One important thing to note is that if we keep predicting forward in time, the heading angle will increase and the linear model will break down. So we usually use linear models for short predictions near the current state.

(d) Now suppose we drive the vehicle from the same starting state, but we turn left instead of going straight, i.e. the steering angle is $\phi[k] = 0.5$ radians. The initial state and input are:

$$\begin{bmatrix} x[0] \\ y[0] \\ \theta[0] \\ v[0] \end{bmatrix} = \begin{bmatrix} 5.0 \\ 10.0 \\ 0.0 \\ 2.0 \end{bmatrix}$$

$$\begin{bmatrix} a[k] \\ \phi[k] \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.5 \end{bmatrix}$$

You can use these values in the IPython notebook to compare how the nonlinear system evolves in comparison to the linear approximation that you made. The IPython notebook simulates the car for ten time steps. Are the trajectories similar or very different? Why?

Solution: No, the linear model we found breaks down in this case, because we have a high steering angle (ϕ = 0.5 radians is a steering angle of roughly 30°). The linear model predicts the same next state as in the previous part. The nonlinear model, on the other hand, has a different prediction after ten time steps.

$$\begin{bmatrix} x[10] \\ y[10] \\ \theta[10] \\ v[10] \end{bmatrix} = \begin{bmatrix} 6.88 \\ 11.3 \\ 1.34 \\ 3 \end{bmatrix}$$

The linear model predicts $[7.45, 10, 0, 3]^T$, and is not close to the prediction of the nonlinear model at all. This is because the linear model has no dependence on $\phi[k]$ — notice the column of zeros in the matrix B. Since $\phi[k] = 0.5$ is very different from zero, the approximation breaks down. To approximate this better we would need to approximate sine and tangent better. In the plot, we notice the nonlinear model predicts the car will turn left. The linear model still predicts that the car will go straight.

3. Image Stitching

Learning Objective: *This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.*

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It's your job to figure out how to stitch the images together using Marcela's common points to reconstruct the larger image.**

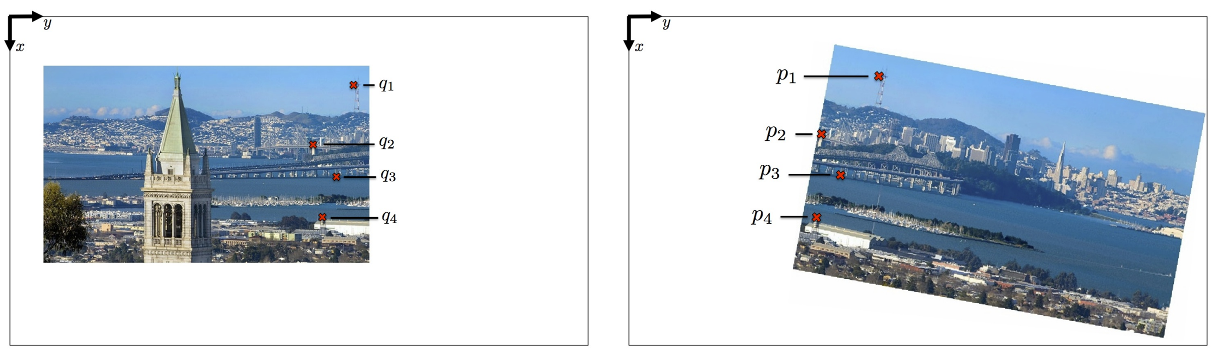


Figure 4: Two images to be stitched together with pairs of matching points labeled.

We will use vectors to represent the common points which are related by a linear transformation. Your idea is to find this linear transformation. For this you will use a single matrix, $\mathbf{R}$, and a vector, $\vec{t}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $\mathbf{R}$ and $\vec{t}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image , which is transformed to $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$ is the corresponding point in the other image (i.e., they represent the same object in the scene). For example, Fig. 4 shows how the points $\vec{p}_1$, $\vec{p}_2$ ... in the right image are transformed to points $\vec{q}_1$, $\vec{q}_2$ ... on the left image. You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\vec{t}} \tag{5}$$

This problem focuses on finding the unknowns (i.e. the components of $\mathbf{R}$ and $\vec{t}$), so that you will be able to stitch the image together.

(a) To understand how the matrix $\mathbf{R}$ and vector $\vec{t}$ transforms any vector representing a point on a image, Consider this equation similar to Equation (5),

$$\vec{v} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u} + \vec{w} = \vec{v}_1 + \vec{w}. \tag{6}$$

Use $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for this part.
We want to find out what geometric transformation(s) can be applied on $\vec{u}$ to give $\vec{v}$.

Step 1: Find out how $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is transforming $\vec{u}$. Evaluate $\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u}$.
What **geometric transformation(s)** might be applied to $\vec{u}$ to get $\vec{v}_1$? Choose the option that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation
(iv) Rotation and Scaling

Drawing the vectors $\vec{u}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations. You can also look into the corresponding demo in the IPython notebook `prob3.ipynb`.
Step 2: Find out $\vec{v} = \vec{v}_1 + \vec{w}$. Find out how **addition of $\vec{w}$ is geometrically transforming** $\vec{v}_1$. Choose the option that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/ Translation.

Drawing the vectors $\vec{v}$, $\vec{w}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations. You can also look into the corresponding demo in the IPython notebook `prob3.ipynb`.
Solution: Plugging in the given vectors and performing the matrix vector multiplication,

$$\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}. \tag{7}$$

It is observable that $\vec{v}_1$ is a scaled, rotated version of $\vec{u}$.
We get $\vec{v} = \vec{v}_1 + \vec{w} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. We can also see that $\vec{v}$ is a shifted version of $\vec{v}_1$.
Hence $\vec{u}$ is scaled, rotated and shifted to get $\vec{v}$.

(b) Multiply Equation (5) out into **two scalar linear equations**.

(i) What are the known values and what are the unknowns in each equation?

(ii) How many unknowns are there?

(iii) How many independent equations do you need to solve for all the unknowns?

(iv) How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns? ***Hint:*** *Remember that each pair of $\vec{p}$ and $\vec{q}$ will give you* ***two*** *different linear equations.*

Solution:

We can rewrite the above matrix equation as the following two scalar linear equations:

$$q_x = p_x r_{xx} + p_y r_{xy} + t_x$$
$$q_y = p_x r_{yx} + p_y r_{yy} + t_y$$

Here, the known values are each pair of points' elements: $q_x,\ q_y,\ p_x,\ p_y$, and 1. The unknowns are elements of $\mathbf{R}$ and $\vec{t}$: $r_{xx},\ r_{xy},\ r_{yx},\ r_{yy},\ t_x$, and t_y. There are 6 unknowns, so we need a total of 6 equations to solve for them. For every pair of points we add, we get two more equations. Thus, we need 3 pairs of common points to get 6 equations.

(c) Write out a system of linear equations that you can use to solve for $\vec{\alpha} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix}$. Assume that all four pairs of points from Fig. 4 are labeled as:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \quad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \quad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix} \quad \vec{q}_4 = \begin{bmatrix} q_{4x} \\ q_{4y} \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} p_{4x} \\ p_{4y} \end{bmatrix}.$$

Now think of your answer to Part b(iv). How many pairs of these points would you need to solve for $\vec{\alpha}$. Choose **just enough** equations required to solve for $\vec{\alpha}$ and express these linear equations using **matrix-vector form**.

Solution: We will choose 3 pairs of points to get 6 equations:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \quad \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \quad \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \quad \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix}$$

The system of linear equations:

$$r_{xx} p_{1x} + r_{xy} p_{1y} + t_x = q_{1x} \tag{8}$$
$$r_{yx} p_{1x} + r_{yy} p_{1y} + t_y = q_{1y} \tag{9}$$
$$r_{xx} p_{2x} + r_{xy} p_{2y} + t_x = q_{2x} \tag{10}$$
$$r_{yx} p_{2x} + r_{yy} p_{2y} + t_y = q_{2y} \tag{11}$$
$$r_{xx} p_{3x} + r_{xy} p_{3y} + t_x = q_{3x} \tag{12}$$
$$r_{yx} p_{3x} + r_{yy} p_{3y} + t_y = q_{3y} \tag{13}$$

We write the system of linear equations in matrix form.

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{1x} & p_{1y} & 0 & 1 \\ p_{2x} & p_{2y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{2x} & p_{2y} & 0 & 1 \\ p_{3x} & p_{3y} & 0 & 0 & 1 & 0 \\ 0 & 0 & p_{3x} & p_{3y} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ q_{1y} \\ q_{2x} \\ q_{2y} \\ q_{3x} \\ q_{3y} \end{bmatrix}$$

(d) In the IPython notebook `prob3.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x, p_y, q_x, and q_y for each pair of points into your system of equations to solve for the matrix, $\mathbf{R}$, and vector, $\vec{t}$. The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. **You are NOT responsible for understanding the image stitching code or Marcela's algorithm.**

Solution:

The parameters for the transformation from the coordinates of the first image to those of the second image are $\mathbf{R} = \begin{bmatrix} 1.1954 & .1046 \\ -.1046 & 1.1954 \end{bmatrix}$ and $\vec{t} = \begin{bmatrix} -150 \\ -250 \end{bmatrix}$.

4. Easing into Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills.*

(a) **Show that if the system of linear equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions, then columns of $\mathbf{A}$ are linearly dependent.** Let us use the structure delineated in **Note 4** to approach this proof. This problem has 4 sub-parts and the following is a chart showing the sequential steps we are going to take to approach this proof.

In a text book you might see the steps in a proof written out in the order in the middle column of th table. But when you are building a proof you usually want to go in another order — this is the order of the subparts in this problem.

Proof steps		Corresponding problem sub-parts
1	Write what is known	Sub-part (i)
2	Manipulate what is known	Sub-part (iii)
3	Connecting it up	Sub-part (iv)
4	What is to be shown	Sub-part (ii)

(i) **First Step: write what you know**
Think about the *information we already know* from the problem statement. We know that system of equations, $\mathbf{A}\vec{x} = \vec{b}$, has infinitely many solutions. Infinitely many solutions are hard to work with, but perhaps we can simplify to something that we can work with. If the system has infinite number of solutions, it must have at least ___ distinct solutions (Fill in the blank).
So let us assume that $\vec{u}$ and $\vec{v}$ are two different vectors, both of which are solutions to $\mathbf{A}\vec{x} = \vec{b}$.
Express the sentence above in a mathematical form (Just writing the equations will suffice; no need to take do further mathematical manipulation).
Solution: If the system has infinite number of solutions, it must have at least two distinct solutions.
(Self-grading comment: Do not reduce points if you forgot to write the answer to the blank in your solutions)
$\vec{u}$ and $\vec{v}$ must satisfy:

$$\mathbf{A}\vec{u} = \vec{b}, \quad \mathbf{A}\vec{v} = \vec{b}. \tag{14}$$

$$\vec{u} \neq \vec{v}. \tag{15}$$

(ii) **What we want to show:**
Now consider *what we need to show*. We have to show that the columns of $\mathbf{A}$ are linearly dependent. Let us assume that $\mathbf{A}$ has columns $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$, i.e. $\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix}$. Using the definition of linear dependence from **Note 3 Subsection 3.1.1**, write a mathematical equation that conveys linear dependence of $\vec{c}_1$, $\vec{c}_2$, ..., and $\vec{c}_n$.
Solution: According to the definition of linear dependence:

$$\alpha_1\vec{c}_1 + \alpha_2\vec{c}_2 + \ldots + \alpha_n\vec{c}_n = \vec{0}. \tag{16}$$

where not all α_i's are equal to zero.

(iii) **Manipulating what we know:**
Now let us try to start from the **First step: equations from (i)**, make mathematically logical steps and reach the **What we want to show: equations from (ii)**. Since your answer to (ii) is expressed in terms of the column vectors of $\mathbf{A}$, let us try to express the mathematical equations from (i), in terms of the the column vectors too. For example, we can write

$$\begin{aligned} \mathbf{A}\vec{x} &= \vec{b} \\ \implies \begin{bmatrix} | & | & \cdots & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} &= \vec{b} \\ \implies x_1\vec{c}_1 + x_2\vec{c}_2 + \ldots + x_n\vec{c}_n &= \vec{b} \end{aligned}$$

Notice that x_1,... x_n etc are scalars. Now use your answer to part (i) to repeat the above formulation for distinct solutions $\vec{u}$ and $\vec{v}$. Note that this is proceeding slightly differently from how we did this proof in lecture. This is fine — there are often many correct ways to do a proof.
Solution:

$$\begin{aligned} \mathbf{A}\vec{u} &= \vec{b} \\ \implies \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_n \end{bmatrix} &= \vec{b} \\ \implies u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n\vec{c}_n &= \vec{b} \end{aligned}$$

$$\begin{aligned} \mathbf{A}\vec{v} &= \vec{b} \\ \implies \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} &= \vec{b} \\ \implies v_1\vec{c}_1 + v_2\vec{c}_2 + \ldots + v_n\vec{c}_n &= \vec{b} \end{aligned}$$

(iv) **Connecting it up:**
Now think about how you can mathematically manipulate your answer from part (iii) (**Manipulating what we know**) to **match the pattern** of your answer from part (ii) (**What we want to show**).
Solution: Subtracting the second equation from the first equation in part (iii), we have

$$u_1\vec{c}_1 + u_2\vec{c}_2 + \ldots + u_n - v_1\vec{c}_1 - v_2\vec{c}_2 - \ldots - v_n\vec{c}_n = \vec{b} - \vec{b} \tag{17}$$
$$\implies (u_1 - v_1)\vec{c}_1 + (u_2 - v_2)\vec{c}_2 + \ldots + (u_n - v_n)\vec{c}_n = \vec{0} \tag{18}$$

Let $\alpha_1 = u_1 - v_1$, ..., and $\alpha_n = u_n - v_n$, i.e. $\vec{\alpha} = \vec{u} - \vec{v}$. Here not all α_i's are equal to zero since $\vec{u} \neq \vec{v}$. Hence the mathematical expression from part (ii) (the **Final Step**) is satisfied, i.e. the proof is complete!

(b) **[PRACTICE]** Now try this proof on your own. Similar proofs will also be covered in your discussion section 3A. Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In order to show this, you have to proof the two following statements:

- If a vector $\vec{q}$ belongs in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belong is $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

In summary, you have to proof the problem statement from both directions. Now use the method developed in part (a) to proof these statements.

Solution:

We start with $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Choosing some scalars a_i, we can write this statement out in mathematical form:

$$\begin{aligned}\vec{q} &= a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \\ &= a_1\vec{v}_1 + a_1\vec{v}_2 + -a_1\vec{v}_2 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n \\ &= a_1(\vec{v}_1 + \vec{v}_2) + (-a_1 + a_2)\vec{v}_2 + \cdots + a_n\vec{v}_n\end{aligned}$$

Since a_1, $(-a_1 + a_2)$ etc are all scalars in the above equation, we can decide that $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have finished proving the first statement. So $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
Now, we must show the other direction. Suppose we have some arbitrary $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$. For some scalars b_i:

$$\begin{aligned}\vec{r} &= b_1(\vec{v}_1 + \vec{v}_2) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n \\ &= b_1\vec{v}_1 + (b_1 + b_2)\vec{v}_2 + \cdots + b_n\vec{v}_n.\end{aligned}$$

Thus, we have shown that $\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Therefore, we have $\text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Combining this with the earlier result, the spans are thus the same.

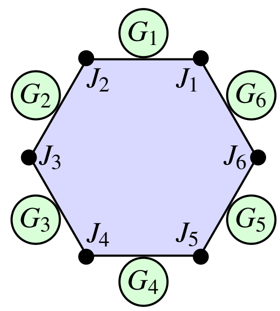
5. Splitting The Tips

Learning Objective: *This problem showcases how you can understand general systems of equations by looking at simpler examples. In particular, see if you can generalize your intuition from the case of 5 and 6 guests to a general number of guests.*

A number of guests gather around a table for a dinner. Between every adjacent pair of guests, there is a jar for tips. When everyone has finished eating, each person places half their tip in the jar to their left and half in the jar to their right. Suppose you can only see the amount of tips in each jar after everyone has left. Can you deduce the amount that each individual tipped?

Note: For this question, if we assume that tips are positive, then we need to introduce additional constraint that would make the system of equations no longer linear. We are going to ignore this constraint and assume that negative tips are acceptable.

(a) Suppose six guests (represented by green circles) sit around a hexagonal table and there are six jars of tips (represented by black dots). If we know the amount of tip in each jar, J_1 to J_6, can we determine each individual's tip amount, G_1 to G_6? If yes, explain why by examining the relationship between the jar values, J_1 to J_6, and guest tips, G_1 to G_6. If not, give two different assignments of G_1 to G_6 that will result in the same J_1 to J_6.



Solution:

No, this is not possible. There exists multiple solutions for G_1 to G_6 given J_1 to J_6. For example, the following two different assignments of tip amounts for each guest:

$$(G_1, G_2, G_3, G_4, G_5, G_6) = (2, 0, 2, 0, 2, 0)$$
$$(G_1, G_2, G_3, G_4, G_5, G_6) = (0, 2, 0, 2, 0, 2)$$

will both result in $(J_1, J_2, J_3, J_4, J_5, J_6) = (1, 1, 1, 1, 1, 1)$.

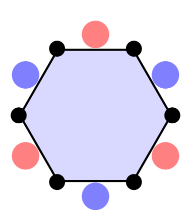
We have the following linear equations for all the jars:

$$\frac{G_1}{2} + \frac{G_6}{2} = J_1, \quad \frac{G_1}{2} + \frac{G_2}{2} = J_2, \quad \frac{G_2}{2} + \frac{G_3}{2} = J_3, \quad \frac{G_3}{2} + \frac{G_4}{2} = J_4, \quad \frac{G_4}{2} + \frac{G_5}{2} = J_5, \quad \frac{G_5}{2} + \frac{G_6}{2} = J_6.$$

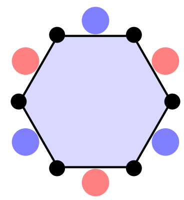
If we write down the system of linear equations in matrix-vector form, we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \\ G_6 \end{bmatrix} = \begin{bmatrix} 2J_1 \\ 2J_2 \\ 2J_3 \\ 2J_4 \\ 2J_5 \\ 2J_6 \end{bmatrix}$$

We can use Gaussian elimination to reduce the last row to all zeros. Therefore, the equations are linearly dependent. Intuitively, we can color guest's each spot at the table with alternating red or blue dots:

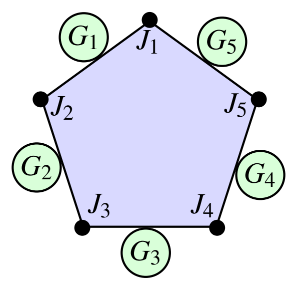


Then supposing that everyone sitting at red spots all tip r dollars and everyone sitting at blue spots all tip b dollars, we find $J_1, \cdots, J_6$ dollars on the jars. However, this situation is no different from the situation illustrated by the following coloring:



Thus, we see that because of the symmetry of the six-sided table, it is not possible to deduce everyone's tip.

(b) Now let's consider five guests around a pentagonal table, G_1 to G_5, and we can see the amount of tips in the five jars, J_1 to J_5. In this new setting can you figure out each guest's tip values, G_1 to G_5?



Solution:

Yes. Our solution will use Gaussian elimination to conclude that the system of equations is linearly independent and a single solution exists. We start by formulating the system of equations as a matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{bmatrix} = \begin{bmatrix} 2J_1 \\ 2J_2 \\ 2J_3 \\ 2J_4 \\ 2J_5 \end{bmatrix}$$

We can run Gaussian elimination on this matrix. In the following order apply the Gaussian elimination operations:

i. subtract row 1 from row 2
ii. subtract row 2 from row 3
iii. subtract row 3 from row 4
iv. subtract row 4 from row 5
v. divide row 5 by a factor of 2

The resulting matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{bmatrix} = \begin{bmatrix} 2J_1 \\ 2J_2 - 2J_1 \\ 2J_3 - 2J_2 + 2J_1 \\ 2J_4 - 2J_3 + 2J_2 - 2J_1 \\ J_5 - J_4 + J_3 - J_2 + J_1 \end{bmatrix} \tag{19}$$

Then in any order apply the Gaussian elimination operations:

i. add row 5 to row 4
ii. subtract row 5 from row 3
iii. add row 5 to row 2
iv. subtract row 5 from row 1

The results is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{bmatrix} = \begin{bmatrix} -J_5 + J_4 - J_3 + J_2 + J_1 \\ J_5 - J_4 + J_3 + J_2 - J_1 \\ -J_5 + J_4 + J_3 - J_2 + J_1 \\ J_5 + J_4 - J_3 + J_2 - J_1 \\ J_5 - J_4 + J_3 - J_2 + J_1 \end{bmatrix} \tag{20}$$

Because we have a full set of pivots columns, a single unique solution exists for G_1 to G_5 in terms of J_1 to J_5.

(c) **[CHALLENGE/OPTIONAL]** *This part will challenge you to further reason about and generalize the results you obtained in parts a and b.*

If n is the total number of guests sitting around a table, for which values of n can you figure out everyone's tip? You do not have to rigorously prove your answer. (**Hint:** consider what is different between parts a and b.)

Solution:

Note: Although you didn't need to prove your answers rigorously, we will give you a rigorous argument. As long as your answer has the flavor of an argument (or an another equally sound argument), you should give yourself full credit.

For even n, this is not possible. Here is a counterexample: Suppose that all the jars had \$1 in them. There are two different ways that this could have happened.

First:

$$G_n = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Second:

$$G_n = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$$

Both cases will result in $J_i = 1$ for all i from 1 to n. Therefore, we cannot figure out everyone's tip.

For odd n, we can determine everyone's tip. Your solution can either argue with Gaussian elimination on a general $n \times n$ matrix where n is odd or use the second argument, which does not use Gaussian elimination:

Gaussian elimination solution:

For odd n, the matrix encoding the system of linear equations is:

$$\begin{array}{cl} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix} & \begin{array}{l} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } n-2 \\ \text{Row } n-1 \\ \text{Row } n \end{array} \end{array}$$

We want to perform Gaussian elimination on this matrix. First, we subtract each row from the one below it until row $n-1$. That is, we sequentially subtract row 1 from row 2, then row 2 from row 3, and so on and so forth until we subtract row $n-1$ from row n.

Let's see what the matrix looks like after the 1^{st} subtraction ($R_2 - R_1$):

$$\begin{array}{cl} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & -1 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix} & \begin{array}{l} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } n-2 \\ \text{Row } n-1 \\ \text{Row } n \end{array} \end{array}$$

Then after the 2^{nd} subtraction ($R_3 - R_2$):

$$\begin{array}{cl} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 1 & 0 & \cdots & 1 \\ & & \ddots & \ddots & & \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix} & \begin{array}{l} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } n-2 \\ \text{Row } n-1 \\ \text{Row } n \end{array} \end{array}$$

And after the $n-2$ subtraction ($R_{n-1} - R_{n-2}$):

$$\begin{array}{cl} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 1 & 0 & \cdots & 1 \\ & & \ddots & \ddots & & \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix} & \begin{array}{l} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } n-2 \\ \text{Row } n-1 \\ \text{Row } n \end{array} \end{array}$$

Can you see the pattern this sequence of subtractions generates? It creates all 1's in the diagonal, alternating 1's and -1's in n^{th} column, and 0's everywhere else. The main thing to notice is that we have -1's on the even rows of the n^{th} column and 1's in the odd rows of the n^{th} column. Since n is odd it follows that $n-1$ will be even and hence we have a -1 in the $a_{n-1,n}$ element of the row reduced matrix.

Finally, we are ready to perform the last subtraction ($R_{n-1} - R_{n-2}$):

$$\begin{array}{cl} \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 1 & 0 & \cdots & 1 \\ & & \ddots & \ddots & & \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 2 \end{bmatrix} & \begin{array}{l} \text{Row } 1 \\ \text{Row } 2 \\ \vdots \\ \text{Row } n-2 \\ \text{Row } n-1 \\ \text{Row } n \end{array} \end{array}$$

We have reached an upper triangular form, which means that the system has a unique solution! To find it, we divide row n by a factor of 2. Lastly, we add row n back to all even rows and subtract it from all odd rows, thus performing the back-substituting step of the Gaussian elimination algorithm.

Alternate solution:
Suppose that each customer tipping: $G_1 = a_1, \cdots, G_n = a_n$ gives rise to the amount in jars $J_1 = j_1, \cdots, J_n = j_n$. This means that we have a solution that looks like $[G_1, G_2, \cdots, G_n] = [a_1, a_2, \cdots, a_n]$, and produces the following amount of tips in the jars: $[J_1, J_2, \cdots, J_n] = [j_1, j_2, \cdots, j_n]$. Now suppose that there exists a different solution that gives rise to exactly the same amount in each and every jar; that is, we still have $[J_1, J_2, \cdots, J_n] = [j_1, j_2, \cdots, j_n]$, but at least one of the tips that a guest has given is different, so at least one of a_i's will be different. Without loss of generality we can assume that this "new" solution will differ for the amount the first guest tips. Also without loss of generality we can choose that $G_1' = a_1 + \varepsilon$. Then, writing out again the second equation of the system we will have:

$$\begin{aligned} G_1' + G_2' &= 2j_2 \\ G_2' &= 2j_2 - G_1' \\ &= (a_1 + a_2) - (a_1 + \varepsilon) \\ &= a_2 - \varepsilon \end{aligned}$$

Imagine now, that we keep writing out our equations one-by-one for every jar. What will happen to the individual tips of all guests? Under the assumption that the amount of tips $[j_1, j_2, \cdots, j_n]$ is kept the same in each and every jar, the individual tips of all the guests will change to become $G_i' = a_i - \varepsilon$ for i even, and $G_i' = a_i + \varepsilon$ for i odd. So, we will have $G_n' = a_n + \varepsilon$ for n odd. In that case the first equation in our system becomes:

$$\begin{aligned} G_1' + G_n' &= 2j_1 \\ G_1' &= 2j_1 - G_n' \\ &= (a_1 + a_n) - (a_n + \varepsilon) \\ &= a_1 - \varepsilon \end{aligned}$$

Which contradicts our initial assumption of $G_1' = a_1 + \varepsilon$, if $\varepsilon \neq 0$!! This means that ε must be zero and we indeed have a unique solution when n is odd. Therefore, everyone's tip can be deduced.

Following this line of reasoning, convince yourself that if a solution exists when n is even, there necessarily must exist infinite solutions as well, and therefore we cannot uniquely deduce every guest's tip.

6. Multiply the Matrices

Learning Objective: *Practice evaluating matrix-matrix multiplication.*

(a) We have two matrices $\mathbf{A}$ and $\mathbf{B}$, where $\mathbf{A}$ is a 3×2 matrix and $\mathbf{B}$ is a 2×4 matrix. Would the multiplication $\mathbf{AB}$ be a valid operation? If yes, what do you expect the dimensions of $\mathbf{AB}$ to be?

Solution:
This is a valid matrix-matrix multiplication as the number of columns in $\mathbf{A}$ matches the number of rows in $\mathbf{B}$. The resulting matrix will have dimensions 3×4, i.e. it will have the same number of rows as $\mathbf{A}$ and the same number of columns as $\mathbf{B}$.

(b) Compute $\mathbf{AB}$ by hand, where $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

Compute **BA** too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

Solution:

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1\times 1+0\times -3 & 1\times 2+0\times 0 & 1\times -1+0\times 2 & 1\times 0+0\times -1 \\ 2\times 1+1\times -3 & 2\times 2+1\times 0 & 2\times -1+1\times 2 & 2\times 0+1\times -1 \\ 0\times 1+1\times -3 & 0\times 2+1\times 0 & 0\times -1+1\times 2 & 0\times 0+1\times -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -3 & 0 & 2 & -1 \end{bmatrix}
\end{aligned}$$

BA does not exist since the number of columns in **B** is not equal to the number of rows in **A**.

(c) Now let us assume $\mathbf{A} \in \mathbb{R}^{2\times n}$ is a **new matrix with 2 rows**, which are given by the **transposes** of column vectors $\vec{r}_1$, $\vec{r}_2$ i.e.

$$\mathbf{A} = \begin{bmatrix} - & \vec{r}_1^{\,T} & - \\ - & \vec{r}_2^{\,T} & - \end{bmatrix} \qquad \text{where,} \qquad \vec{r}_1 = \begin{bmatrix} r_{11} \\ r_{12} \\ \vdots \\ r_{1n} \end{bmatrix}, \text{ and } \qquad \vec{r}_2 = \begin{bmatrix} r_{21} \\ r_{22} \\ \vdots \\ r_{2n} \end{bmatrix}.$$

$\mathbf{B} \in \mathbb{R}^{n\times 3}$ is a **new matrix with 3 columns**, which are called $\vec{c}_1$, $\vec{c}_2$, and $\vec{c}_3$, i.e.

$$\mathbf{B} = \begin{bmatrix} | & | & | \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ | & | & | \end{bmatrix} \qquad \text{where,} \qquad \vec{c}_1 = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1n} \end{bmatrix}, \qquad \vec{c}_2 = \begin{bmatrix} c_{21} \\ c_{22} \\ \vdots \\ c_{2n} \end{bmatrix}, \text{ and } \qquad \vec{c}_3 = \begin{bmatrix} c_{31} \\ c_{32} \\ \vdots \\ c_{3n} \end{bmatrix}$$

Now show that:

$$\mathbf{AB} = \begin{bmatrix} \vec{r}_1^{\,T}\vec{c}_1 & \vec{r}_1^{\,T}\vec{c}_2 & \vec{r}_1^{\,T}\vec{c}_3 \\ \vec{r}_2^{\,T}\vec{c}_1 & \vec{r}_2^{\,T}\vec{c}_2 & \vec{r}_2^{\,T}\vec{c}_3 \end{bmatrix}$$

if **AB** is a valid operation.

Solution:

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} - & \vec{r}_1^{\,T} & - \\ - & \vec{r}_2^{\,T} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \\ | & | & | \end{bmatrix} \\
&= \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ \vdots & \vdots & \vdots \\ c_{1n} & c_{2n} & c_{3n} \end{bmatrix} \\
&= \begin{bmatrix} r_{11}c_{11}+\ldots+r_{1n}c_{1n} & r_{11}c_{21}+\ldots+r_{1n}c_{2n} & r_{11}c_{31}+\ldots+r_{1n}c_{3n} \\ r_{21}c_{11}+\ldots+r_{2n}c_{1n} & r_{21}c_{21}+\ldots+r_{2n}c_{2n} & r_{21}c_{31}+\ldots+r_{2n}c_{3n} \end{bmatrix}
\end{aligned}$$

Since we can write $r_{11}c_{11} + \ldots + r_{1n}c_{1n} = \vec{r_1}^T\vec{c_1}$, $r_{11}c_{21} + \ldots + r_{1n}c_{2n} = \vec{r_1}^T\vec{c_2}$ etc, we can substitute these values in the expression above to get:

$$\mathbf{AB} = \begin{bmatrix} \vec{r_1}^T\vec{c_1} & \vec{r_1}^T\vec{c_2} & \vec{r_1}^T\vec{c_3} \\ \vec{r_2}^T\vec{c_1} & \vec{r_2}^T\vec{c_2} & \vec{r_2}^T\vec{c_3} \end{bmatrix}.$$

(d) The following matrix is an example of a special type of matrix called a nilpotent matrix.

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

What happens to this matrix when you you raise it to some power, i.e. multiply it by itself repeatedly? Let us find out! Calculate $\mathbf{C}^3$ by hand. Make sure you show what $\mathbf{C}^2$ is along the way.
(Just for thought: Why do you think this is called a "nilpotent" matrix? Of course, there is nothing magical about a 3×3 matrix. You can have nilpotent square matrices of any dimension greater than 1.)
Solution: A nilpotent matrix is a matrix that becomes all 0's when you raise it to some power, i.e. repeatedly multiply it by itself.

$$\begin{aligned} \mathbf{C}^2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \mathbf{C}^3 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

For the purposes of this homework, the above is all you need to know. If you are interested to know more about this nilpotent matrix, you are welcome to read the following, which is **out of scope** for 16A!
(*This particular matrix can be used to differentiate a second order polynomial* $u + vx + wx^2$, *i.e.* $\mathbf{C}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} v \\ 2w \\ 0 \end{bmatrix}$ *can be used analogously to* $\frac{d}{dx}[u + vx + wx^2] = v + 2wx$.

For example,

$$\frac{d}{dx}[1] = 0$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\frac{d}{dx}[x] = 1$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\frac{d}{dx}\left[x^2\right] = 2x$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

*We can differentiate $1+x+x^2$ once by multiplying with **C** once:*

$$\frac{d}{dx}\left[1+x+x^2\right] = 1+2x$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

*We can differentiate twice by applying **C** twice:*

$$\frac{d^2}{dx^2}\left[x^2+x+1\right] = 2$$
$$\mathbf{C}^2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

and then thrice:

$$\frac{d^3}{dx^3}\left[x^2+x+1\right] = 0$$
$$\mathbf{C}^3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is consistent with the fact that applying the third derivative to a second order polynomial will always yield a result of 0!)

7. Exam Policy and Practice

Please read through the entirety of the EECS16A exam proctoring policy (click here) carefully before proceeding. This question is designed to familiarize you with how the exam will be run and help you setup and practice.

(a) After reading through the Exam Policy carefully, please answer the following questions.

i. Given you experience no disruptions during the exam, how many total minutes do you have for scanning and submission? Does it matter if you are using a tablet or whether you are using paper to write your answers? What if there is a disruption?

ii. Are you required to record locally during the exam? How much space should you have available on your computer for a local recording?

iii. How should you contact the course staff in case of an emergency situation during the exam?

Solution:

i. You have a total of 20 minutes for scanning and submission if you experience no disruption and are using paper. 10 minutes if you are using a tablet. If you experience x minutes of disruption during the exam, you may work for $\min(x, 15)$ minutes past the end of the exam.

ii. You are not required to record locally; you may do a Zoom cloud recording. You should have 5 GB available on your computer if you are doing a local recording.

iii. You should contact the course staff by making a private post on Piazza. However, you should not be looking at Piazza during the exam other than to make a private post in the case of an emergency.

(b) Please configure your Zoom link.

i. Fill out the following Google Form (click here) to submit the Zoom link you will be using. You must use this Zoom link for this assignment as well as for the exams. If you wish, you may use your Personal Meeting Room link and set your Personal Meeting ID as your default on all devices (desktop + laptop + phone) you will be using for the exams.

ii. Ensure that anyone can join your Zoom link and that there is no waiting room for your Zoom meeting. Try to do this by entering the meeting on one device that is logged in to your official Berkeley Zoom account and then entering the meeting on another device that is logged into some other Zoom account. If you are able to join automatically, then your Zoom link is *joinable*. If you are not put into a waiting room, then your Zoom meeting will not have a waiting room. (You might want to have a member of your study group try this out with you if you don't already have two Zoom accounts.)

Solution: Ensure your Zoom link is *joinable* and that the Zoom link in the form is the correct link which you will be using for the exam.

(c) You will now practice a Zoom recording. You should use this recording to work through a homework problem or other study material to simulate the actual circumstances of the final exam.

i. Start the Zoom call for the link you provided above. Turn on your microphone and recording device (webcam, phone camera). You may turn off your speaker. Please share your entire desktop (not just a particular window). Your video should be visible on the desktop and at maximum size. Please refer to the image below.

ii. Start recording via Zoom. You may record locally or on the cloud (see the exam policy document for details).

iii. Hold your CalID or any photo ID (as detailed in the exam policy document) next to your face and record yourself saying your name into the webcam. Both your face and your entire CalID should be visible in the video. We should be able to read your name and SID. This step should take **at least** 3 seconds. See Fig. 5.

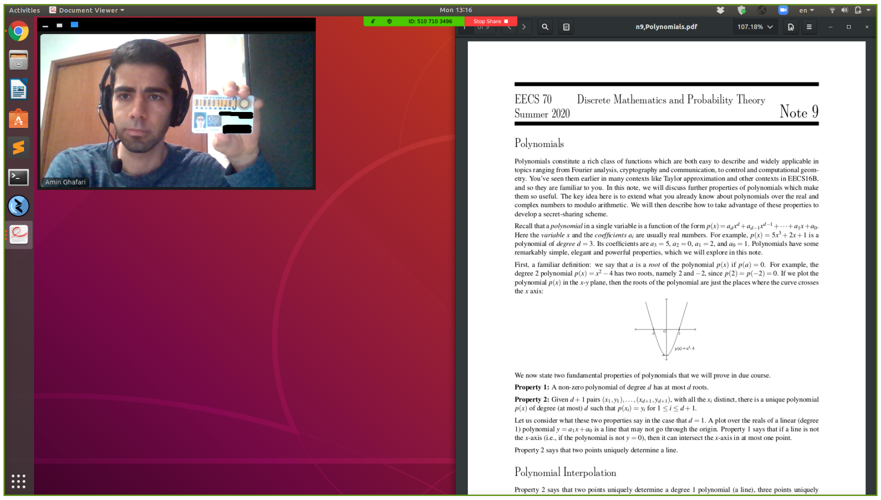


Figure 5: ID card demonstration. Do not black out your SID and name.

iv. Turn your recording device (webcam, phone) around 360° **slowly** so that we can see your entire room clearly. There should be no uncovered screens anywhere in the room during your exam. Only admin TAs and instructors will be able to see your videos (both for this assignment and for the actual exams).

v. Position your recording device in such a way that we can see your workspace and your hands. It is perfectly fine if your face is not visible at this point. If you are not using your phone to record your workspace, then it should be visible in the recording, face down. See figure 3. **Think about how you want to set this up during this test run. On the actual exam, you will want to use the computer to see the exam itself, so make sure this works for you. If you are using a laptop's built in webcam to record and also see the exam, make sure you have a setup that works for you. Think about how you might position the laptop. You may also consider using your phone or an external webcam on a stand to record your workspace. We want you to iron out these details ahead of the actual exam so that the exam itself has no additional stress due to proctoring logistics. Please contact STEP if you would like to request an external webcam.**

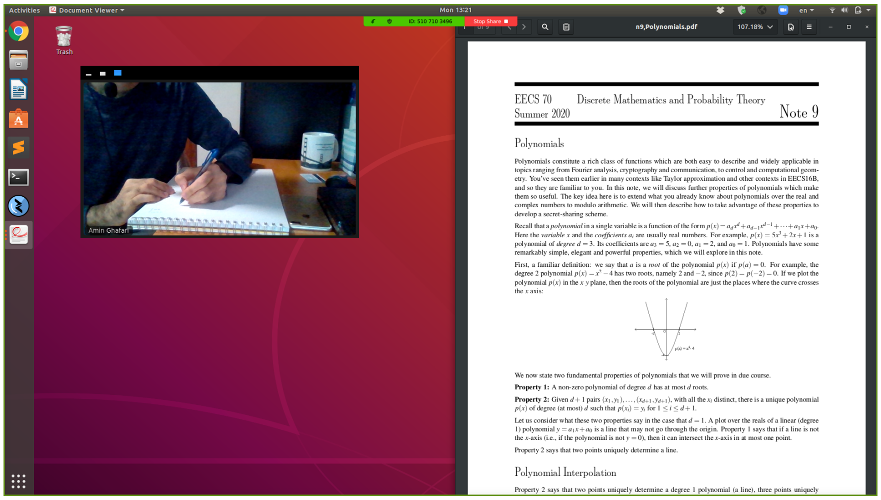


Figure 6: Demonstration of taking your exam. Your setup should look like this while you are taking the exam.

vi. Your microphone should be on at all times. The recording should also include the time on your desktop at all times.

vii. Record for a full two and a half hours. You should use this time to work through a homework problem or other study material for the course. The more realistic it is to actually taking an exam, the better practice it will be for you. (We also want to make sure that your computer can handle the video encoding task if your are doing a local recording.)

viii. After two and a half hours, make sure your face is back in your recording and then stop the recording. In the actual exam, prior to ending the recording you will scan and submit your exam to Gradescope.

ix. After stopping the recording, check your recording to confirm that it contains your video as well as your desktop throughout its duration. Upload your video to Google drive (if saved locally) and submit a link to the video using this Google Form (click here). Please name your file using the studentname-exam naming format described in the proctoring instructions. Please make sure that the link sharing option gives us viewing permission.

Solution: Ensure that you were able to submit your video to the Google form.

(d) A Midterm Google document should be shared with you with instructions for the midterm a few days before the midterm. More details will be made available closer to the exam date.

Link for policy:
`https://docs.google.com/document/d/1EVb4Ca6FWSAykExY7X5ynFW4KdmHd0BI6KZ0ktM8ows/edit?usp=sharing`

Form to submit Zoom link:
`https://forms.gle/QjnfMoA2L6hMkuVQA`

Form to submit video link:
`https://forms.gle/eb8D1weGca1WyMDh7`

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.