EECS 16A Designing Information Devices and Systems I
Fall 2020 Homework 12

This homework is due November 20, 2020, at 23:59.
Self-grades are due November 23, 2020, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw12.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit the file to the appropriate assignment on Gradescope.

1. Reading Assignment

For this homework, please read Notes 21 and 22 to learn about inner products, norms, trilateration, and correlation. It may also be helpful to recap Note 19 about op-amps, negative feedback, and common amplifiers designs. You are always encouraged to read beyond this as well.

(a) What does it mean for two vectors $\vec{x}$ and $\vec{y}$ to be orthogonal, in terms of their inner product?

Solution: The inner product of orthogonal vectors is zero, $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \, \|\vec{y}\| \cos 90^o = \|\vec{x}\| \, \|\vec{y}\| \cdot 0 = 0$

(b) In trilateration, the distances between the beacons and the unknown location $\vec{x}$ involve quadratic terms of $\vec{x}$. What trick can we use to get a system of linear equations in $\vec{x}$?

Solution: We can get a system of linear equations by subtracting one non-linear equation from another and eliminating the quadratic terms.

(c) Suppose the signal $x[n]$ is only defined for timesteps $0, 1, \cdots, 5$. For the purpose of computing linear cross-correlation, what value of $x[n]$ do we assume when n is a timestep out of the range: $0 \leq n \leq 5$ (e.g. $n = 6$ or $n = -1$)?

Solution: When n is a timestep out of the range: $0 \leq n \leq 5$, we consider $x[n]$ to be zero.

2. Island Karaoke Machine

(Contributors: Aviral Pandey, Lam Nguyen, Pangiotis Zarkos, Sashank Krishnamurthy, Titan Yuan, Urmita Sikder, Vijay Govindarajan)

Learning Goal: *The objective of this problem is design a circuit that calculates the difference between two signals and amplifies the result.*

You're stuck on a desert island and everyone is bored out of their minds. Fortunately, you have your EECS16A lab kit with op-amps, wires, resistors, and your handy breadboard. You decide to build a karaoke machine. You recover one speaker from the crash remains and use your iPhone as your source. You know that many songs put instruments on either the "left" or the "right" channel, but the vocals are usually present on both channels with equal strength.

In our case, the vocals are present on both left and right channels, but the instruments are only present on the right channel, i.e.

$$v_{\text{left}} = v_{\text{vocals}}$$
$$v_{\text{right}} = v_{\text{vocals}} + v_{\text{instrument}},$$

where the voltage source v_{vocals} can have values anywhere in the range of $\pm 120\,\text{mV}$ and $v_{\text{instrument}}$ can have values anywhere in the range of $\pm 50\,\text{mV}$.

What is the goal of a karaoke machine? **The ultimate goal is to *remove* the vocals from the audio output.** We're going to do this by first building a circuit that takes the left and right audio outputs of the smartphone and then calculates its **difference**. Let's see what happens.

The equivalent circuit model of the iPhone audio jack and speaker is shown in Figure 1. We model the **audio signals and jack** as v_{left} and v_{right} with **equivalent source resistance** of the left/right audio channels of $R_{\text{left}} = R_{\text{right}} = 3\,\Omega$. The **speaker** has an equivalent resistance of $R_{\text{speaker}} = 4\,\Omega$.

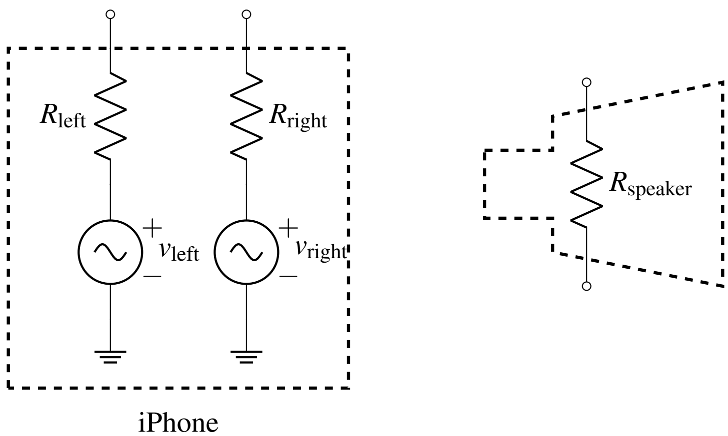


Figure 1: Audio jack and speaker of an iPhone.

(a) One of your island survivors suggests the circuit in Figure 2 to do this. **Find the expression for the voltage across the speaker R_{speaker} as a function of v_{vocals} and $v_{\text{instruments}}$.**
Does the voltage across the speaker depend on v_{vocals}? In other words, what do you think the islanders will hear – vocals, instruments, or both?

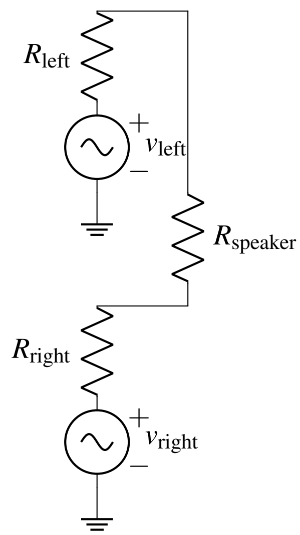
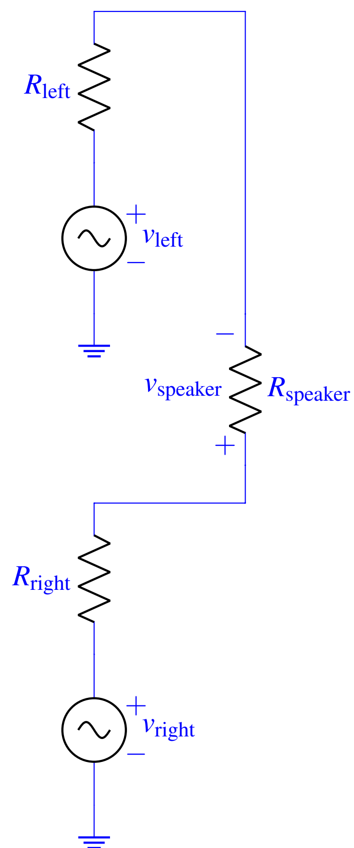


Figure 2: Circuit for part (a).

Solution:

Let's mark the voltage across the speaker, v_{speaker}, from bottom to top as in the figure:



We can apply the principle of superposition to solve for v_{speaker}. First, we solve for the voltage across the speaker when only v_{left} is on. Let's call this $v_{\text{speaker,left}}$. Notice that the circuit becomes a voltage divider. Therefore, we get

$$-v_{\text{speaker,left}} = \frac{v_{\text{left}} R_{\text{speaker}}}{R_{\text{speaker}} + R_{\text{left}} + R_{\text{right}}} = \frac{4v_{\text{vocals}}}{10} = 0.4v_{\text{vocals}},$$

giving

$$v_{\text{speaker,left}} = -0.4v_{\text{vocals}}.$$

Similarly, we solve for the voltage across the speaker when only v_{right} is on. Let's call this $v_{\text{speaker,right}}$. Again, notice that the circuit becomes a voltage divider. Therefore, we get

$$v_{\text{speaker,right}} = \frac{v_{\text{right}} R_{\text{speaker}}}{R_{\text{speaker}} + R_{\text{left}} + R_{\text{right}}} = \frac{4(v_{\text{vocals}} + v_{\text{instrument}})}{10} = 0.4(v_{\text{vocals}} + v_{\text{instrument}}).$$

Superposition tells us that $v_{\text{speaker}} = v_{\text{speaker,left}} + v_{\text{speaker,right}} = 0.4v_{\text{instrument}} = 0.4 \cdot 50\,\text{mV} = 20\,\text{mV}$. What did you notice? The vocals got canceled out! The islanders will only hear the instruments, just as they wanted.

(b) We need to boost the sound level to get the party going. To this end, we want a range of $\pm 2\,\text{V}$ across the speaker. Design a circuit by completing the Figure 3 below that takes in $\{v_{\text{left}}, R_{\text{left}}\}$ and $\{v_{\text{right}}, R_{\text{right}}\}$ combos as inputs and outputs an **amplified version of $v_{\text{instrument}}$ across R_{speaker}**. Consider all op-amps to be **ideal** for this problem.

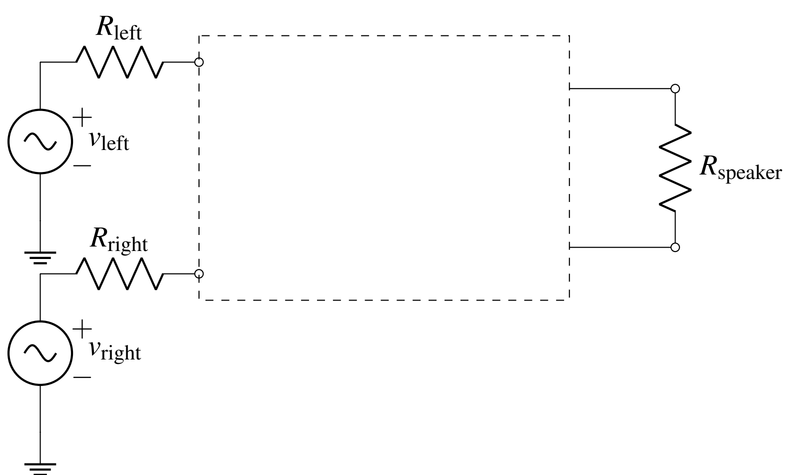


Figure 3: Circuit for part (b).

Hint 1: *We need to get an output voltage with the range of $\pm 2\,\text{V}$. The input voltage $v_{instrument}$ can have values anywhere in the range of $\pm 50\,\text{mV}$. What gain is needed from the op-amp based amplifier circuits?*

Hint 2: *Use two op-amps in the non-inverting configuration. The non-ideal voltage source $\{v_{left}, R_{left}\}$ must be the input to one non-inverting amplifier and the non-ideal voltage source $\{v_{right}, R_{right}\}$ must be the input to the other non-inverting amplifier.*

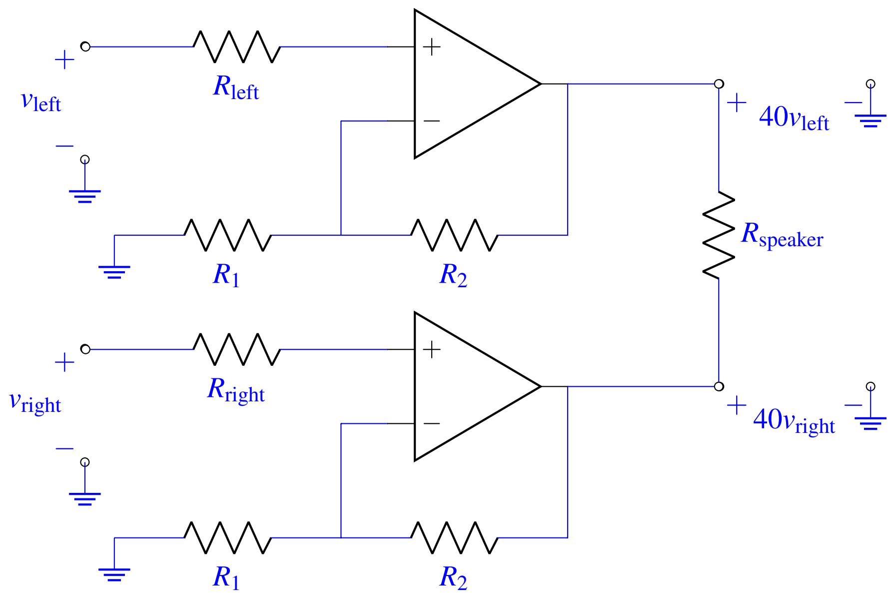
Hint 3: *Connect the speaker $R_{speaker}$ across the outputs of those two op-amps.*

Solution:

Feed the non-ideal voltage source $\{v_{\text{left}}, R_{\text{left}}\}$ into a non-inverting amplifier with gain A_v and the non-ideal voltage $\{v_{\text{right}}, R_{\text{right}}\}$ into another non inverting amplifier with gain A_v. (We have a different gain from the previous part, which we need to determine.) Then connect the two outputs across R_{speaker} as shown in the previous part.

In this circuit, we will get $v_{\text{speaker}} = A_v \cdot v_{\text{instrument}}$. Since $v_{\text{instrument}}$ has a range of $\pm 50\,\text{mV}$, v_{speaker} will have a range of $\pm 50\,\text{mV} \cdot A_v = \pm 0.05 \cdot A_v \text{V}$. Now we need $\pm 0.05 \cdot A_v \text{V} = \pm 2$, i.e. $A_v = 40$.

Therefore, we want to design a non-inverting amplifier with voltage gain of 40 using the circuit shown below:



Now, we need to find R_1 and R_2.

$$A_v = 1 + \frac{R_2}{R_1}$$

Therefore, we can then choose any R_1 and R_2 such that $\frac{R_2}{R_1} = 39$. Note that there are multiple ways of choosing them. One such choice is $R_1 = 1\,\text{k}\Omega$ and $R_2 = 39\,\text{k}\Omega$, for instance.

(c) The trouble with the approach in part (b) is that two op-amps are required. Let's say you only have **one op-amp** with you. What would you do? One night in your dreams, you have an inspiration. Why not combine the inverting and non-inverting amplifier into one, as shown in Figure 4!

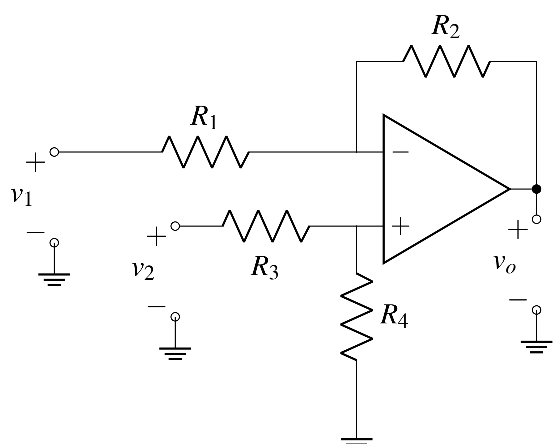


Figure 4: The new amplifier for part (c).

If we set $v_2 = 0\,\text{V}$, what is the output v_o in terms of v_1? (This is the inverting path.)

Solution:

If we set $v_2 = 0\,\text{V}$, we would get $u_+ = 0\,\text{V}$. Applying the Golden Rules, we will get $u_- = u_+ = 0\,\text{V}$. Writing KCL at the $-$ terminal of the op-amp, we get

$$\frac{v_1 - 0}{R_1} = \frac{0 - v_{o,1}}{R_2},$$

which gives

$$v_{o,1} = \frac{-v_1 R_2}{R_1}.$$

(d) Consider the circuit in Figure 4 again. If we set $v_1 = 0\,\text{V}$, what is the output v_o in terms of v_2? (This is the non-inverting path.)

Solution:

If we set $v_1 = 0\,\text{V}$, we would get $u_+ = \frac{v_2 R_4}{R_3 + R_4} = u_-$. Writing KCL at the $-$ terminal gives

$$\frac{0 - u_-}{R_1} = \frac{u_- - v_{o,2}}{R_2},$$

which gives

$$v_{o,2} = u_-\left(1 + \frac{R_2}{R_1}\right) = v_2\left(\frac{R_4}{R_3 + R_4}\right)\left(1 + \frac{R_2}{R_1}\right).$$

(e) Now, determine v_o in terms of v_1 and v_2 by superposing your results from the last two parts. Choose values for R_1, R_2, R_3 and R_4, such that the speaker has $v_o = \pm 2\,\text{V}$ across it for $v_2 - v_1 = \pm 50\,\text{mV}$.

Solution:

By the principle of superposition,

$$v_o = v_{o,1} + v_{o,2}.$$

If we set $v_1 = v_{\text{left}}$ and $v_2 = v_{\text{right}}$, we'd ideally want $v_o = -40v_1 + 40v_2$. We can choose R_1, R_2, R_3 and R_4, so that this happens.

How do we do this? Let's do this in steps. First, note that, looking for the expression for $v_{o,1}$, we'll want $\frac{R_2}{R_1} = 40$. Therefore, we can choose any values of R_2 and R_1, such that this happens. One such choice is $R_1 = 1\,\text{k}\Omega$ and $R_2 = 40\,\text{k}\Omega$. Then, plug that into the expression of $v_{o,2}$, and the condition we now want is

$$\left(\frac{R_4}{R_3 + R_4}\right)\left(1 + \frac{R_2}{R_1}\right) = 40,$$

which gives us

$$\frac{R_4}{R_3 + R_4} = \frac{40}{41}.$$

Thus, we need to choose R_3 and R_4. As before, we can choose these values in many ways. One such choice is $R_4 = 40\,\text{k}\Omega$ and $R_3 = 1\,\text{k}\Omega$.

Note: Keep in mind that, for this problem, we actually assumed that $v_1 = v_{\text{left}}$ and $v_2 = v_{\text{right}}$, which would mean that we are ideally connecting v_{left} and v_{right} as inputs. However, in reality, we're actually connecting the outputs from the iPhone as inputs. This means that R_{left} and R_{right} will also actually affect the output.

With this effect, we will actually get

$$v_o = -\frac{v_1 R_2}{R_{1,eq}} + v_2\left(\frac{R_4}{R_{3,eq} + R_4}\right)\left(1 + \frac{R_2}{R_{1,eq}}\right),$$

where $R_{1,eq} = R_1 + R_{\text{left}}$ and $R_{3,eq} = R_3 + R_{\text{right}}$.

Therefore, we can just *fold in* the effect of R_{left} and R_{right} into these. For instance, we want to set $R_{3,eq} = 1\,\text{k}\Omega$. Now, we can actually make $R_3 = R_{3,eq} - 3\,\Omega = 997\,\Omega$ and $R_1 = R_{1,eq} - 3\,\Omega = 997\,\Omega$.

Give yourself full credit even if you didn't notice this, but keep this in mind!

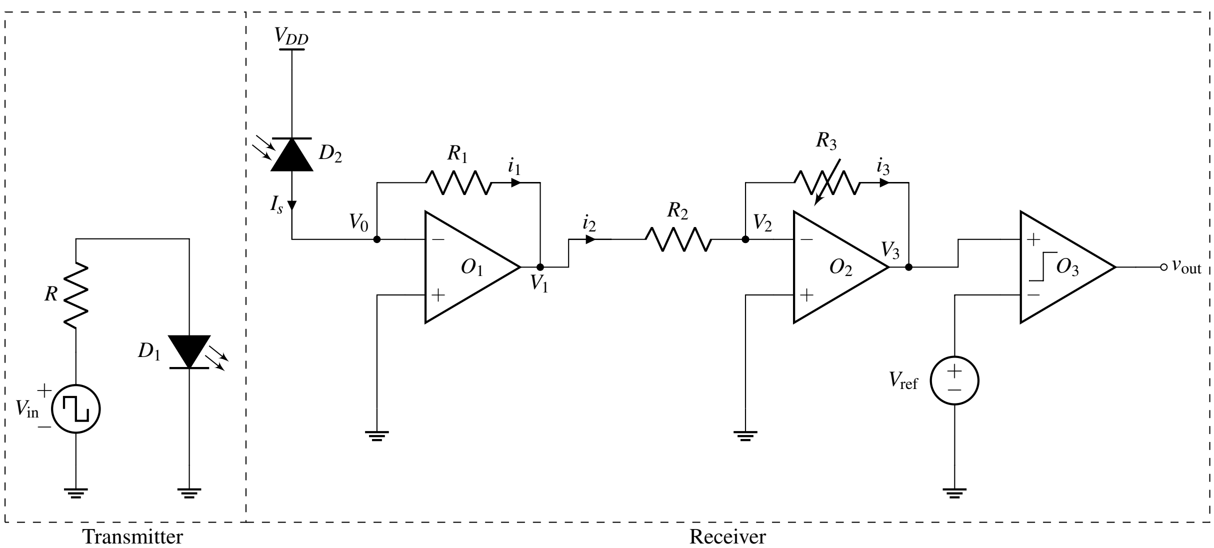
Bonus: Can you now see why we wanted to keep R_1 and R_3 in the order of kΩ or larger?

3. Wireless Communication With An LED

(Contributors: Aviral Pandey, Lam Nguyen, Pangiotis Zarkos, Sashank Krishnamurthy, Titan Yuan, Urmita Sikder)

Learning Goal: *This problem is designed to find the response of a complex circuit by breaking it into smaller circuit blocks and analyzing each block individually.*

In this question, we are going to analyze the system shown in the figure below. It shows a circuit that can be used as a wireless communication system using visible light (or infrared, very similar to remote controls).



The element D_1 in the transmitter is a ***light-emitting diode (LED).*** An LED is an element that emits light and whose brightness is controlled by the current flowing through it. You can recall controlling the light emitted by an LED using your MSP430 in Touchscreen Lab 1. In our circuit, the current through the LED, hence its brightness, can be controlled by choosing the applied voltage V_{in} and the value of the resistor R.

The light from the element D_1 in the transmitter in the transmitter is received by the element D_2 in the receiver. In this circuit, the LED D_1 is used as a means for **transmitting information with light**, and the element D_2 is used as **a light receiver to see if anything was transmitted**.

In the receiver, the element labeled as D_2 behaves like a ***"reverse biased" solar cell***, which means that when it receives light, it generates a current. We will denote the current generated by the solar cell by I_S.

Remark: *In Imaging Lab 3, we talked about how non-idealities, such as background light, affect the performance of a system that does light measurements. In this question, we will assume ideal conditions, that is, there is no source of light around except for the LED.*

In our system, we define two states for the transmitter: (i) the transmitter is sending something when **they turn on the LED** and (ii) the transmitter is not sending anything when **they turn off the LED**. On the receiver side, the goal is to **convert the current I_S generated by the solar cell into a voltage and amplify it**, so that we can read the output voltage V_{out} to see if the transmitter was sending something or not. The circuit implements this operation through a series of op-amps and a comparator. *It might look look complicated at first glance, but we can analyze it one section at a time.*

(a) Currents i_1, i_2 and i_3 are labeled on the diagram. Assuming *op-amps are ideal* and the Golden Rules hold, is $I_S = i_1$? $i_1 = i_2$? $i_2 = i_3$? Treat the solar cell as an ideal current source supplying I_s.

Solution:

We use the Golden Rules, which say that in an op-amp, no current flows into or out of V_+ or V_-. Therefore we can use KCL at node V_0 and V_2 to conclude that $\boxed{I_S = i_1}$ and $\boxed{i_2 = i_3}$. However, if $I_S \neq 0$ and $R_1 \neq R_2$, then $\boxed{i_1 \neq i_2}$. This is because $V_0 = V_2 = 0\,\text{V}$ and V_1 is some non-zero voltage. If $R_1 \neq R_2$, then the currents flowing through them are different.

(b) Use the Golden Rules to find V_0, V_1, V_2 and V_3 in terms of I_S, R_1, R_2 and R_3.

Hint: Solve for V_0 first, then use V_0 to find V_1. Afterwards, use V_1 to find V_2; and use V_2 to find V_3.

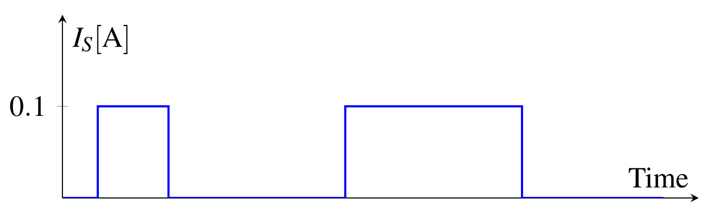
Solution:

Using the Golden Rules, we know that $\boxed{V_0 = 0\,\text{V}}$. Using Ohm's law, we know that $V_0 - V_1 = i_1 R_1$. From the previous part, we know that $i_1 = I_S$. Thus, we get $\boxed{V_1 = -I_S R_1}$. Using the Golden Rules again, we get $\boxed{V_2 = 0\,\text{V}}$. Using Ohm's law and the KCL result from the previous part, we get the following equations:

$$V_1 - V_2 = i_2 R_2$$
$$V_2 - V_3 = i_3 R_3$$
$$i_2 = i_3$$

Solving them, we get $\boxed{V_3 = I_S \frac{R_1 R_3}{R_2}}$.

(c) Now, assume that the transmitter has chosen the values of V_{in} and R to control the intensity of light emitted by LED, such that when the **transmitter is sending something**, I_S is equal to 0.1 A and when the **transmitter is not sending anything**, I_S is equal to 0 A. The following figure shows a visual example of how this current I_S might look like as time changes.



For the receiver, suppose $V_{ref} = 2\,\text{V}$, $R_1 = 10\,\Omega$, $R_2 = 1000\,\Omega$, and the supply voltages of the comparator are $V_{DD} = 5\,\text{V}$ and $V_{SS} = -5\,\text{V}$. Pick a value of R_3 such that **V_{out} is V_{DD} when the transmitter is sending something (i.e. $I_S = 0.1\,\text{A}$) and V_{SS} when the transmitter is not sending anything (i.e. $I_S = 0$).**

Solution:

We want $v_{out} = V_{DD}$, when $V_3 - V_{ref} > 0\,\text{V}$ when $I_S = 0.1\,\text{A}$ and $v_{out} = V_{SS}$, when $V_3 - V_{ref} < 0\,\text{V}$ when $I_S = 0\,\text{A}$. We plug the known resistor values into the equation in the previous part to get $V_3 = I_S \frac{R_3}{100}$.

When $I_S = 0\,\text{A}$, $V_3 - V_{\text{ref}} = 0\,\text{V} - 2\,\text{V} < 0\,\text{V}$, so $v_{\text{out}} = V_{SS} = -5\,\text{V}$. When $I_S = 0.1\,\text{A}$, $V_3 - V_{\text{ref}} = 0.1 \times \frac{10R_3}{1000} - V_{\text{ref}} = \frac{R_3}{1000} - 2\,\text{V} > 0\,\text{V}$, so $\frac{R_3}{1000} > 2\,\text{V}$. Thus, $\boxed{R_3 > 2000\,\Omega}$. For any $R_3 > 2000\,\Omega$, $V_3 - V_{\text{ref}} > 0\,\text{V}$, so $v_{\text{out}} = V_{DD} = 5\,\text{V}$.

4. (Optional/Practice) Otamatone Stars (Fa18, MT2)

(Contributors: Alan Zhang, Ryan Tsang, Sarika Madhvapathy, Urmita Sikder, Vijay Govindarajan)

Learning Goal: *The objective of this problem is to explore circuit design by building small functional circuit blocks and putting them together.*

Alan and Ryan are trying to become YouTube musical sensations by playing the ***otamatone***, which is a funny looking instrument shaped like a music note (Figure 5). With one hand you press your fingers to the touch pad on the stem of the otamatone, and with the other hand you squeeze open its mouth, which plays the sound corresponding to the position of your finger. **As the position of your finger moves up the stem (away from the mouth) the frequency of the sound gets higher.**

Instead of buying an otamatone, Ryan thinks it's cooler to build one. He has a spare "Black Box Speaker System" that takes a voltage input V_{BB}, and outputs a sound, as shown in Figure 6.

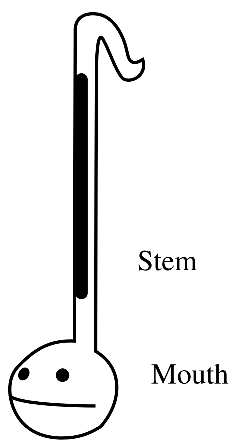


Figure 5: An Otamatone

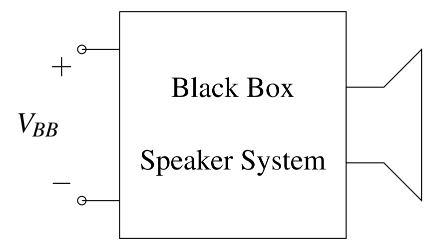


Figure 6: Black Box Speaker System

All that he is missing is an input for the black box, but you notice that you can design the pad on the stem of the otamatone exactly like you designed the **one-dimensional resistive touchscreen**, as seen in Figure 7.

We are given the following information:

- **Increasing V_{BB} increases the frequency of the output sound.**
- The Black Box Speaker requires a minimum input voltage of $V_{BB,min} > 0$ and outputs its lowest frequency at the input voltage, $V_{BB} = V_{BB,min}$.
- The darkly shaded resistive bars in Figure 7 are identical with a cross sectional area A, uniform resistivity ρ, and length L.
- The variable x in Figure 7 indicates the **position of touch.**
- **We need to design an otamatone such that as we touch closer to the top ($x = L$), output sound with higher frequency is produced.**

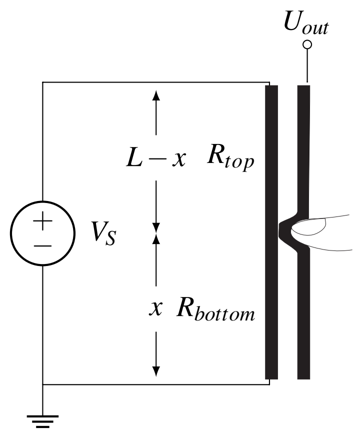


Figure 7: Resistive Touchscreen Schematic

(a) Referring to Figure 7, find the resistances R_{bottom} and R_{top} in terms of x and other physical parameters. R_{bottom} is the resistance of the **left resisitive bar** below the finger touch point x. R_{top} is the resistance of the left resistive bar above the touch point.

Solution:
We can find both these values by simply using the equation for resistance of a physical structure: $R = \frac{\rho L}{A}$, where L is the length of the corresponding portion of the resistive bar and A is the area cross section.

$$R_{bottom} = \frac{\rho x}{A}$$

$$R_{top} = \frac{\rho(L-x)}{A}$$

(b) Find an expression for U_{out}, when there is touch, in terms of V_S, R_{top} and R_{bottom}. *Hint: You might start by drawing the equivalent circuit diagram.*

Solution:
We can apply the voltage divider equation to find:

$$U_{out} = V_S \frac{R_{bottom}}{R_{top} + R_{bottom}}$$

(c) Consider the range of U_{out} for the full range of touch positions: $0 \leq x \leq L$. Can we directly plug our U_{out} into our black box speaker system if we want to utilize **the full range of our touchscreen**? Recall that the Black Box requires a minimum input voltage of $V_{BB,min} > 0$ to function.

Solution:
No. The minimum value of U_{out} is 0V (corresponding to $x = 0$), and Ryan's black box needs at least $V_{BB,min}$.

(d) Rishi and Sarika see that you are working hard on your design. To help you out, they give you the circuit block in Figure 8 to consider:

Find V_{out} in terms of V_{in} and other parameters. What does the circuit do? Consider the op-amp to be ideal.

Solution:
The circuit shown is a voltage summer, so $V_{out} = V_{in} + V_{BB,min}$.

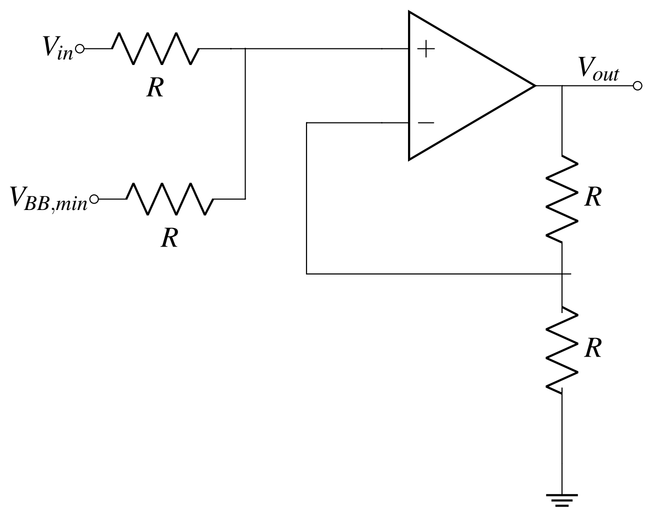


Figure 8: Circuit for part (d)

One solution approach is as follows.

Applying Golden Rule 1, we note that V_+ is the output of a voltage divider:

$$V_+ = V_{BB,min} + \frac{R}{2R}(V_{in} - V_{BB,min}) = \frac{V_{in} + V_{BB,min}}{2}$$

Next, we observe the op-amp is in a negative feedback configuration, so we can apply Golden Rule 2:

$$V_+ = V_- = \frac{V_{in} + V_{BB,min}}{2}$$

Finally, applying Golden Rule 1 again, we note that V_- is the output of a voltage divider, so:

$$V_- = \frac{R}{2R} V_{out} \Rightarrow V_{out} = 2V_-$$

$$V_{out} = V_{in} + V_{BB,min}$$

.

(e) Now, complete your otamatone design! **Complete the figure below** by drawing the connections between the components that the circuit blocks function as intended. Recall that you want to increase frequency of the speaker system as you move your finger toward the top of the otamatone stem. You may use **only one additional op-amp and one additional voltage source** and no other components.

Hint: Consider the loading effect if you connect U_{out} to the input of the op-amp circuit from part (d). Also keep in mind the speaker's minimum input voltage requirement in your design.

Solution:

First, we realize if we directly connect the touchscreen to the voltage summer, there will be loading effects as U_{out} will change due the input resistors of the summer. So we first connect a unity gain buffer at the output of the touchscreen to decouple the touchscreen from the summer circuit. As V_{out} is the output of an op-amp, no buffer is needed after the summer.

The full circuit will look like such: [Touchscreen] + [Buffer] + [Voltage Summer] + [BB Circuit]

(f) Using your design from Figure 9, find V_{BB}, the signal going into the black box speaker system, as a function of x, the location of touch on the touch bar.

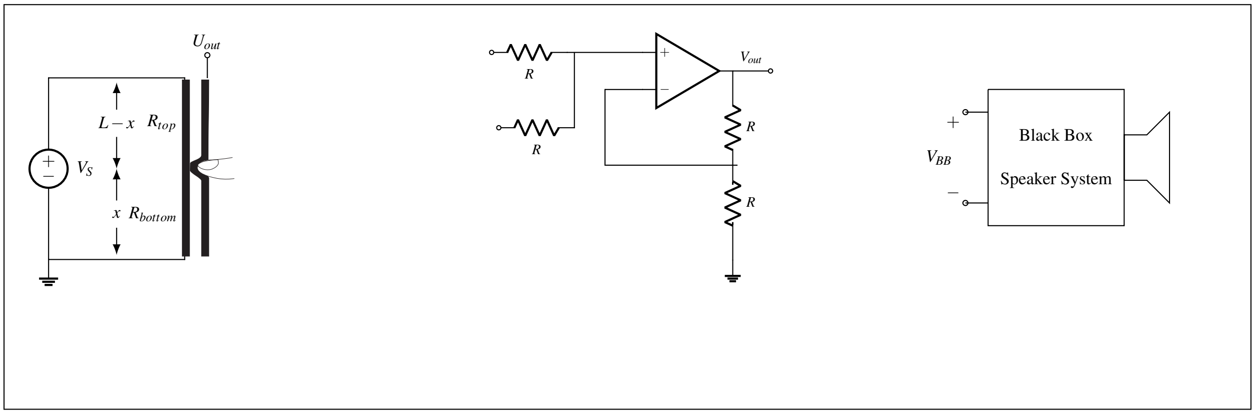


Figure 9: Design for (e)

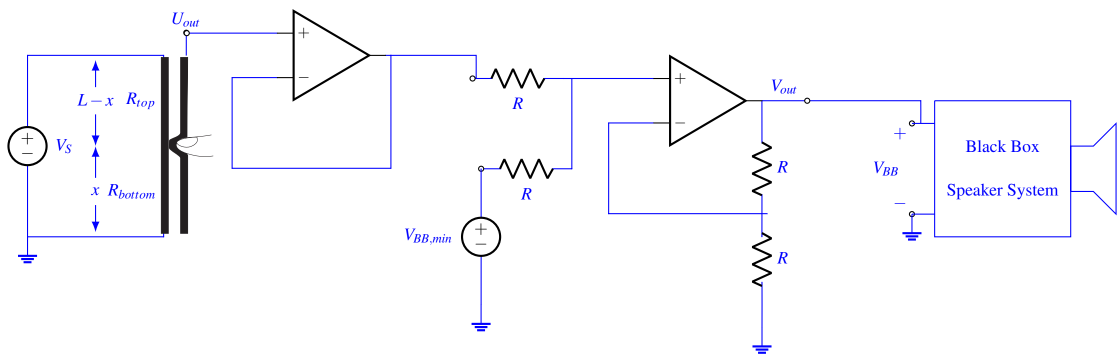


Figure 10: Design solution for part (e)

Solution:
We first compute the touchscreen output, U_{out}:

$$U_{out} = V_S \frac{R_{bottom}}{R_{top} + R_{bottom}} = V_S \frac{\frac{\rho x}{A}}{\frac{\rho(L-x)}{A} + \frac{\rho x}{A}} = V_S \frac{x}{L}$$

We then compute the summer output, V_{out}:

$$V_{out} = U_{out} + V_{BB,min} = V_S \frac{x}{L} + V_{BB,min}$$

Since the output of the summer is fed into the speaker, then $V_{out} = V_{BB}$ so:

$$V_{BB} = V_S \frac{x}{L} + V_{BB,min}$$

This meets our design specifications of using the whole length of the touchscreen to produce different output sound frequencies and of increasing the frequency as we touch closer to the top.

5. Mechanical Linear Correlation

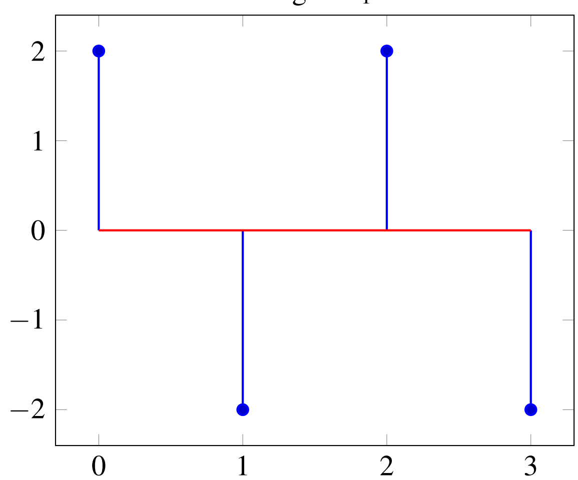
(Contributors: Michael Kellman, Avi Pandey, Linda Liu, Moses Won, Amanda Jackson, Gireeja Ranade, Lam Nguyen, Laura Brink, Sang Min Han, Spencer Kent, Urmita Sikder, Vijay Govindarajan, Raghav Anand)

Learning Goal: *The objective of this problem is to understand how to compute the linear correlation between signals.*

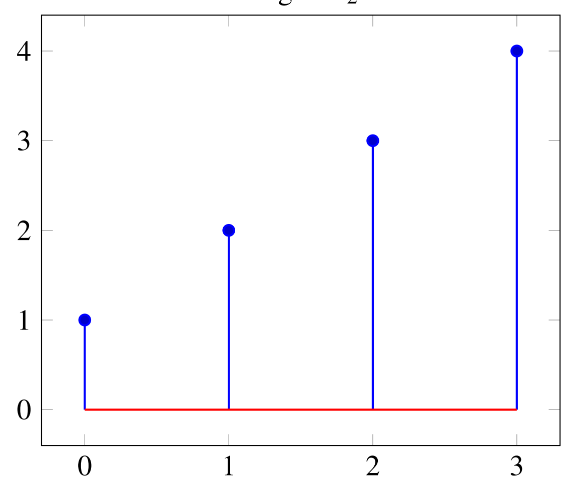
We recall that the linear correlation of signal $\vec{y}$ with signal $\vec{x}$ is given as:

$$\text{corr}_{\vec{x}}(\vec{y})[k] = \sum_{n=-\infty}^{\infty} \vec{x}[n]\vec{y}[n-k]$$

Signal $\vec{s}_1$



Signal $\vec{s}_2$



Assume that both of the above signals extend to $\pm\infty$, and are 0 everywhere outside of the region shown in the above graphs. First, we will demonstrate the procedure for linear correlation by computing the linear correlation between signal $\vec{s}_1$ with itself (*i.e.* $\text{corr}_{\vec{s}_1}(\vec{s}_1)[k]$). This is referred to as the linear autocorrelation. This can be computed by evaluating the inner product between the signal and the shifted version of the signal (outlined in the below tables). Here, we compute this quantity for shifts between -3 and 3. For all shifts outside this range, the inner product is zero. Finally, we plot the non-zero values of the linear autocorrelation.

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n+3]$	2		-2		2		-2		0		0		0		0		0		0	
$\langle \vec{s}_1[n], \vec{s}_1[n+3] \rangle$	0	+	0	+	0	+	-4	+	0	+	0	+	0	+	0	+	0	+	0	= -4

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n+2]$	0		2		-2		2		-2		0		0		0		0		0	
$\langle \vec{s}_1[n], \vec{s}_1[n+2] \rangle$	0	+	0	+	0	+	4	+	4	+	0	+	0	+	0	+	0	+	0	= 8

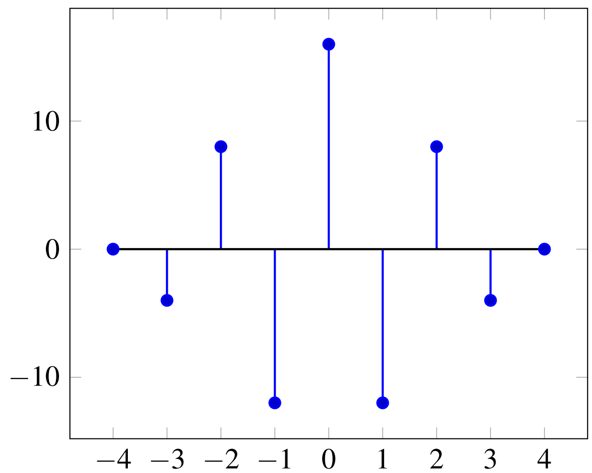
$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n+1]$	0		0		2		-2		2		-2		0		0		0		0	
$\langle \vec{s}_1[n], \vec{s}_1[n+1] \rangle$	0	+	0	+	0	+	-4	+	-4	+	-4	+	0	+	0	+	0	+	0	= -12

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_1[n+0]$	0		0		0		2		-2		2		-2		0		0		0	
$\langle \vec{s}_1[n], \vec{s}_1[n+0] \rangle$	0	+	0	+	0	+	4	+	4	+	4	+	4	+	0	+	0	+	0	= 16

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0	
$\vec{s}_1[n-1]$	0	0	0	0	2	-2	2	-2	0	0	
$\langle \vec{s}_1[n], \vec{s}_1[n-1] \rangle$	0 +	0 +	0 +	0 +	-4 +	-4 +	-4 +	0 +	0 +	0	= -12

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0	
$\vec{s}_1[n-2]$	0	0	0	0	0	2	-2	2	-2	0	
$\langle \vec{s}_1[n], \vec{s}_1[n-2] \rangle$	0 +	0 +	0 +	0 +	0 +	4 +	4 +	0 +	0 +	0	= 8

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0	
$\vec{s}_1[n-3]$	0	0	0	0	0	0	2	-2	2	-2	
$\langle \vec{s}_1[n], \vec{s}_1[n-3] \rangle$	0 +	0 +	0 +	0 +	0 +	0 +	-4 +	0 +	0 +	0	= -4



(a) Using the procedure demonstrated above, compute $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$, the linear cross-correlation of $\vec{s}_2$ with $\vec{s}_1$. Like the example, use tables like the one given below for $k = -3$ and plot the resulting correlation.

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0
$\vec{s}_2[n+3]$	1	2	3	4	0	0	0	0	0	0
$\langle \vec{s}_1[n], \vec{s}_2[n+3] \rangle$										

Solution:

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0	
$\vec{s}_2[n+3]$	1	2	3	4	0	0	0	0	0	0	
$\langle \vec{s}_1[n], \vec{s}_2[n+3] \rangle$	0 +	0 +	0 +	8 +	0 +	0 +	0 +	0 +	0 +	0	= 8

$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0	
$\vec{s}_2[n+2]$	0	1	2	3	4	0	0	0	0	0	
$\langle \vec{s}_1[n], \vec{s}_2[n+2] \rangle$	0 +	0 +	0 +	6 +	-8 +	0 +	0 +	0 +	0 +	0	= -2

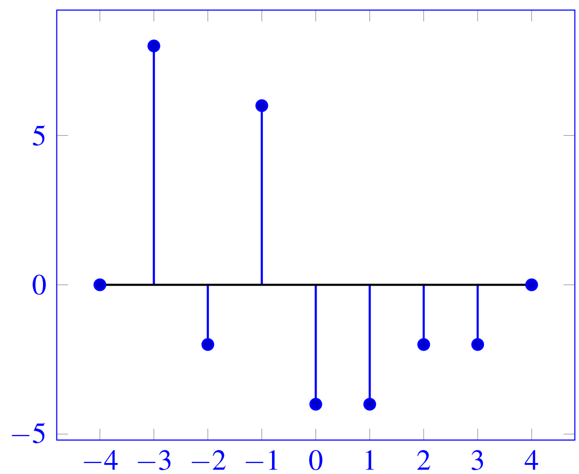
$\vec{s}_1[n]$	0	0	0	2	-2	2	-2	0	0	0	
$\vec{s}_2[n+1]$	0	0	1	2	3	4	0	0	0	0	
$\langle \vec{s}_1[n], \vec{s}_2[n+1] \rangle$	0 +	0 +	0 +	4 +	-6 +	8 +	0 +	0 +	0 +	0	= 6

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_2[n+0]$	0		0		0		1		2		3		4		0		0		0	
$\langle \vec{s}_1[n], \vec{s}_2[n+0] \rangle$	0	+	0	+	0	+	2	+	-4	+	6	+	-8	+	0	+	0	+	0	= -4

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_2[n-1]$	0		0		0		0		1		2		3		4		0		0	
$\langle \vec{s}_1[n], \vec{s}_2[n-1] \rangle$	0	+	0	+	0	+	0	+	-2	+	4	+	-6	+	0	+	0	+	0	= -4

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_2[n-2]$	0		0		0		0		0		1		2		3		4		0	
$\langle \vec{s}_1[n], \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	0	+	2	+	-4	+	0	+	0	+	0	= -2

$\vec{s}_1[n]$	0		0		0		2		-2		2		-2		0		0		0	
$\vec{s}_2[n-2]$	0		0		0		0		0		0		1		2		3		4	
$\langle \vec{s}_1[n], \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	-2	+	0	+	0	+	0	= -2



(b) Will the linear cross-correlation of $\vec{s}_2$ with $\vec{s}_1$ ($\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$) be the same as the cross-correlation of $\vec{s}_1$ with $\vec{s}_2$ ($\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$)? You can use the iPython notebook **prob12.ipynb** to figure this out. How are they related to each other?

Solution: See sol12.ipynb. They do not have the same result, but they are related: one is the reverse of the other. If you were able to observe this, give yourself full points.

You were not explicitly required to show why, but a sketch of why this is the case follows. Let us compare $\text{corr}_{\vec{s}_2}(\vec{s}_1)[k]$ and $\text{corr}_{\vec{s}_1}(\vec{s}_2)[k]$.

By definition:

$$\text{corr}_{\vec{s}_2}(\vec{s}_1)[k] = \sum_{n=-\infty}^{\infty} \vec{s}_2[n]\vec{s}_1[n-k]$$

$$\text{corr}_{\vec{s}_1}(\vec{s}_2)[k] = \sum_{n=-\infty}^{\infty} \vec{s}_1[n]\vec{s}_2[n-k]$$

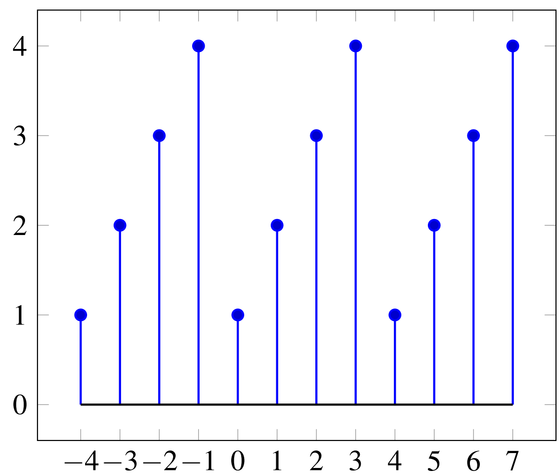
Using a substitution of index, $m = n - k$ we have:

$$
\begin{aligned}
\mathrm{corr}_{\vec{s}_1}(\vec{s}_2)[k] &= \sum_{m=-\infty}^{\infty} \vec{s}_1[m+k]\vec{s}_2[m] \\
&= \sum_{m=-\infty}^{\infty} \vec{s}_2[m]\vec{s}_1[m-(-k)] \\
&= \mathrm{corr}_{\vec{s}_2}(\vec{s}_1)[-k]
\end{aligned}
$$

So we can conclude that $\mathrm{corr}_{\vec{s}_1}(\vec{s}_2)[k] = \mathrm{corr}_{\vec{s}_2}(\vec{s}_1)[-k]$.

Now, we will review the procedure to perform linear cross-correlation between one signal that is periodic with a period of 4 and another that is finite length and extended with zeros as in the previous parts. As an example, we will compute the linear correlation $\mathrm{corr}_{\vec{p}_2}(\vec{s}_1)[k]$ between the periodic signal $\vec{p}_2$ (with period 4), formed by repeating $\vec{s}_2$, and the finite length signal $\vec{s}_1$ extended with zeros. The result will be a periodic signal with period 4.

The periodic signal, $\vec{p}_2$, formed by repeating $\vec{s}_2$ is plotted below for indices -4 to 7. It is defined and non-zero for all indices from $-\infty$ to $+\infty$.



We compute one period of the result of the cross-correlation by starting at a shift of $k = -1$ and ending at a shift of $k = 2$.

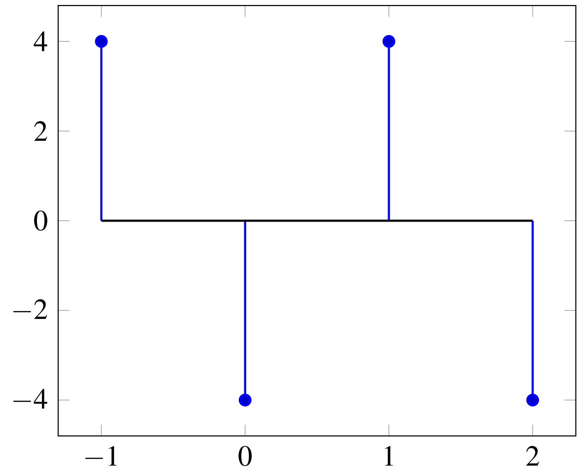
$\vec{p}_2[n]$	2		3		4		1		2		3		4		1		2		3		
$\vec{s}_1[n+1]$	0		0		2		-2		2		-2		0		0		0		0		
$\langle \vec{p}_2[n], \vec{s}_1[n+1] \rangle$	0	+	0	+	8	+	-2	+	4	+	-6	+	0	+	0	+	0	+	0		= 4

$\vec{p}_2[n]$	2		3		4		1		2		3		4		1		2		3		
$\vec{s}_1[n+0]$	0		0		0		2		-2		2		-2		0		0		0		
$\langle \vec{p}_2[n], \vec{s}_1[n+0] \rangle$	0	+	0	+	0	+	2	+	-4	+	6	+	-8	+	0	+	0	+	0		= -4

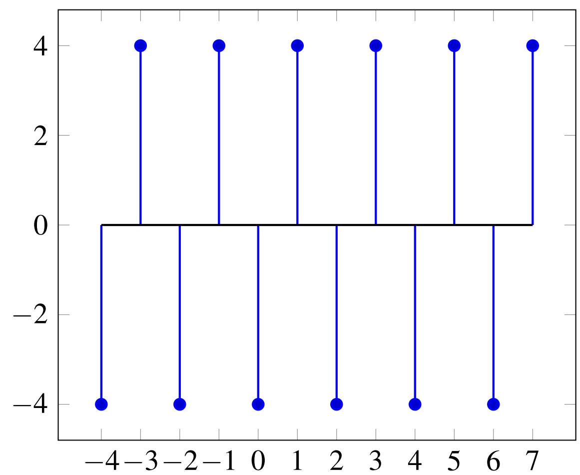
$\vec{p}_2[n]$	2		3		4		1		2		3		4		1		2		3		
$\vec{s}_1[n-1]$	0		0		0		0		2		-2		2		-2		0		0		
$\langle \vec{p}_2[n], \vec{s}_1[n-1] \rangle$	0	+	0	+	0	+	0	+	4	+	-6	+	8	+	-2	+	0	+	0		= 4

$\vec{p}_2[n]$	2		3		4		1		2		3		4		1		2		3		
$\vec{s}_1[n-2]$	0		0		0		0		0		2		-2		2		-2		0		
$\langle \vec{p}_2[n], \vec{s}_1[n-2] \rangle$	0	+	0	+	0	+	0	+	0	+	6	+	-8	+	2	+	-4	+	0		= -4

The computed single period of the resulting linear cross correlation is plotted below.



The resulting linear cross correlation for shifts from $k = -4$ to $k = 7$ is plotted below.



(c) Repeat the procedure described above to compute the correlation $\text{corr}_{\vec{p}_1}(\vec{s}_1)[k]$ between a periodic signal $\vec{p}_1$ (with period 4), formed by repeating $\vec{s}_1$, and the finite-length signal $\vec{s}_1$ extended with zeros. Like the example, evaluate tables like the one below for $k = -3$ for different shifts and plot a single period of the result.

$\vec{p}_1[n]$	-2	2	-2	2	-2	2	-2	2	-2	2
$\vec{s}_1[n+3]$	2	-2	2	-2	0	0	0	0	0	0
$\langle \vec{p}_1[n], \vec{s}_1[n+3] \rangle$										

Solution: We have computed below shifts from $k = -3$ to $k = 3$. However, so long as you have enough values for a single period, give yourself full credit.

$\vec{p}_1[n]$	-2		2		-2		2		-2		2		-2		2		-2		2		
$\vec{s}_1[n+3]$	2		-2		2		-2		0		0		0		0		0		0		
$\langle \vec{p}_1[n], \vec{s}_1[n+3] \rangle$	-4	+	-4	+	-4	+	-4	+	0	+	0	+	0	+	0	+	0	+	0		= -16

$\vec{p}_1[n]$	-2		2		-2		2		-2		2		-2		2		-2		2	
$\vec{s}_1[n+2]$	0		2		-2		2		-2		0		0		0		0		0	
$\langle \vec{p}_1[n], \vec{s}_1[n+2] \rangle$	0	+	4	+	4	+	4	+	4	+	0	+	0	+	0	+	0	+	0	= 16

$\vec{p}_1[n]$	-2		2		-2		2		-2		2		-2		2		-2		2	
$\vec{s}_1[n+1]$	0		0		2		-2		2		-2		0		0		0		0	
$\langle \vec{p}_1[n], \vec{s}_1[n+1] \rangle$	0	+	0	+	-4	+	-4	+	-4	+	-4	+	0	+	0	+	0	+	0	= -16

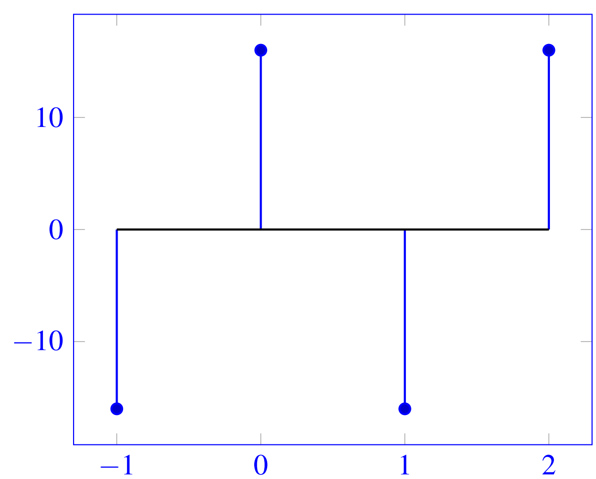
$\vec{p}_1[n]$	-2		2		-2		2		-2		2		-2		2		-2		2	
$\vec{s}_1[n+0]$	0		0		0		2		-2		2		-2		0		0		0	
$\langle \vec{p}_1[n], \vec{s}_1[n+0] \rangle$	0	+	0	+	0	+	4	+	4	+	4	+	4	+	0	+	0	+	0	= 16

$\vec{p}_1[n]$	-2		2		-2		2		-2		2		-2		2		-2		2	
$\vec{s}_1[n-1]$	0		0		0		0		2		-2		2		-2		0		0	
$\langle \vec{p}_1[n], \vec{s}_1[n-1] \rangle$	0	+	0	+	0	+	0	+	-4	+	-4	+	-4	+	-4	+	0	+	0	= -16

$\vec{p}_1[n]$	-2		2		-2		2		-2		2		-2		2		-2		2	
$\vec{s}_1[n-2]$	0		0		0		0		0		2		-2		2		-2		0	
$\langle \vec{p}_1[n], \vec{s}_1[n-2] \rangle$	0	+	0	+	0	+	0	+	0	+	4	+	4	+	4	+	4	+	0	= 16

$\vec{p}_1[n]$	-2		2		-2		2		-2		2		-2		2		-2		2	
$\vec{s}_1[n-3]$	0		0		0		0		0		0		2		-2		2		-2	
$\langle \vec{p}_1[n], \vec{s}_1[n-3] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	-4	+	-4	+	-4	+	-4	= -16

Like the example, the period was plotted from $k = -1$ to $k = 2$. Give yourself full credit if you plotted four consecutive values sufficient for a single period, i.e. your plot starts from a shift of $k = k_0$ and ends at $k = k_0 + 3$



6. Cauchy-Schwarz Inequality

(Contributors: Amanda Jackson, Avi Pandey, Gireeja Ranade, Michael Kellman, Panos Zarkos, Richard Liou, Vijay Govindarajan, Titan Yuan)

Learning Goal: *The objective of this problem is to understand and prove the Cauchy-Schwarz inequality for real-valued vectors.*

The Cauchy-Schwarz inequality states that for two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$|\langle \vec{v}, \vec{w} \rangle| = |\vec{v}^T \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

In this problem we will prove the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^2$.

Take two vectors: $\vec{v} = r \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\vec{w} = t \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$, where $r > 0$, $t > 0$, θ, and ϕ are scalars. Make sure you understand why any vector in $\mathbb{R}^2$ can be expressed this way and why it is acceptable to restrict $r, t > 0$.

(a) In terms of some or all of the variables r, t, θ, and ϕ, what are $\|\vec{v}\|$ and $\|\vec{w}\|$?

Solution: We use the trig identity $\cos^2 x + \sin^2 x = 1$ to show:

$$\begin{aligned} \|\vec{v}\| &= \sqrt{v_1^2 + v_2^2} \\ &= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= r \end{aligned}$$

Similarly, $\|\vec{w}\| = t$.

(b) In terms of some or all of the variables r, t, θ, and ϕ, what is $\langle \vec{v}, \vec{w} \rangle$?

Solution: We use the trig identity $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x-y)$ to show:

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= (r\cos\theta)(t\cos\phi) + (r\sin\theta)(t\sin\phi) \\ &= r \cdot t(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ &= r \cdot t\cos(\theta - \phi) \end{aligned}$$

(c) Show that the Cauchy-Schwarz inequality holds for any two vectors in $\mathbb{R}^2$. *Hint: consider your results from part (b). Also recall* $-1 \leq \cos x \leq 1$ *and use both inequalities.*

Solution: We use the fact that $\cos x \leq 1$ to show:

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= r \cdot t\cos(\theta - \phi) \\ &= \|\vec{v}\|\|\vec{w}\|\cos(\theta - \phi) \\ &\leq \|\vec{v}\|\|\vec{w}\| \end{aligned}$$

We use the fact that $\cos x \geq -1$ to show:

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= r \cdot t\cos(\theta - \phi) \\ &= \|\vec{v}\|\|\vec{w}\|\cos(\theta - \phi) \\ &\geq -\|\vec{v}\|\|\vec{w}\| \end{aligned}$$

Therefore:

$$-\|\vec{v}\|\|\vec{w}\| \leq \langle \vec{v}, \vec{w} \rangle \leq \|\vec{v}\|\|\vec{w}\|,$$

which gives us that

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\|\|\vec{w}\|.$$

(d) Note that the inequality states that the inner product of two vectors must be less than *or equal to* the product of their magnitudes. What conditions must the vectors satisfy for the equality to hold? In other words, when is $\langle\vec{v},\vec{w}\rangle = \|\vec{v}\|\cdot\|\vec{w}\|$?

Solution:

$$\begin{aligned}\langle\vec{v},\vec{w}\rangle &= \|\vec{v}\|\|\vec{w}\| \\ \|\vec{v}\|\|\vec{w}\|\cos(\theta-\phi) &= \|\vec{v}\|\|\vec{w}\| \\ \cos(\theta-\phi) &= 1 \\ \theta-\phi &= 0\end{aligned}$$

We see that the equality holds when the angle between the two vectors is zero. Note that when the angle is zero, the vectors would be linearly dependent.

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

Solution:

I first worked by myself for 2 hours, but got stuck on problem 5. Then I met with my study group.

XYZ played the role of facilitator ... etc. We were still stuck on problem 5 so we went to office hours to talk about the problem.

Then I went to homework party for a few hours, where I finished the homework.