EECS16A DIS 14B

Last discussion!

Learning Objectives

① Least squares special case: Orthonormal columns of A in $A\vec{x} \approx \vec{b}$

(We looked @ Orthogonal columns last time)

↳ $\text{proj}_{C(A)} \vec{b} = \text{proj}_{\vec{a}_1} \vec{b} + \text{proj}_{\vec{a}_2} \vec{b} + \dots + \text{proj}_{\vec{a}_m} \vec{b}$

ⓐ Orthonormality buys us not having to add extra correction terms. Will explain more. (in the least squares context)

ⓑ Orthonormality can give us linear independence (in general)

② Correlation: checking presence of signals by using correlation/inner product as a similarity measure

Music

Thundercat - Them Changes

Procol Harum - Whiter Shade of Pale

Corinne Bailey Rae - Paris Nights/New York mornings

EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 14B

$$I \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \qquad I \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

1. **Orthonormal Matrices and Projections**

An orthonormal matrix, $\mathbf{A}$, is a matrix whose columns, $\vec{a}_i$, are:

- Orthogonal (ie. $\langle \vec{a}_i, \vec{a}_j \rangle = 0$ when $i \neq j$)
- Normalized (ie. vectors with length equal to 1, $\|\vec{a}_i\| = 1$). This implies that $\|\vec{a}_i\|^2 = \langle \vec{a}_i, \vec{a}_i \rangle = 1$.

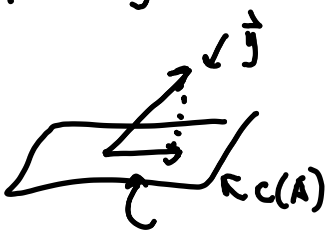
(a) Suppose that the matrix $\mathbf{A} \in \mathbb{R}^{N\times M}$ has linearly independent columns. The vector $\vec{y}$ in $\mathbb{R}^N$ is not in the subspace spanned by the columns of $\mathbf{A}$. What is the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{A}$?

$$A = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_M \\ | & | & & | \end{bmatrix}}_{M \text{ cols}} \qquad \vec{a}_i \in \mathbb{R}^N \quad \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{Ni} \end{bmatrix} \qquad \vec{y} \in \mathbb{R}^N$$

$\vec{y} \notin C(A)$

$\notin \text{Span}\{\vec{a}_1, \dots, \vec{a}_M\}$

$\min_{\vec{x}} \|A\vec{x} - \vec{y}\|^2$



$\text{proj}_{C(A)} \vec{y} = A \cdot \text{least squares solution}$

$= A(A^TA)^{-1}A^T\vec{y}$

A has lin indep. cols $\Rightarrow (A^TA)^{-1}$ exists

$M\times M$
$N\times N$
$\downarrow N\times M$

(b) Show if $\mathbf{A} \in \mathbb{R}^{N\times N}$ is an orthonormal matrix then the columns, $\vec{a}_i$, form a basis for $\mathbb{R}^N$.

Show this: If $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_N\}$ is a basis

1. $\{\vec{a}_1, \dots, \vec{a}_N\}$ is linearly independent
2. $\text{Span}\{\vec{a}_1, \dots, \vec{a}_N\} = \mathbb{R}^N$

$N(A) = N(A^TA)$

A has lin indep. cols

$\Rightarrow N(A) = \{\vec{0}\}$

$\Rightarrow N(A^TA) = \{\vec{0}\}$

$\Rightarrow A^TA$ has lin indep cols

$(A^TA)^{-1}$ exists

1. Want to show $c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots c_N\vec{a}_N = \vec{0}$

$\Rightarrow 0 = c_1 = c_2 = c_3 = \cdots = c_N$

if $c_1\vec{a}_1 + \cdots + c_N\vec{a}_N = \vec{0}$

$\langle \vec{a}_i, c_1\vec{a}_1 + \cdots + c_N\vec{a}_N \rangle = \langle \vec{a}_i, \vec{0} \rangle = 0$

$\langle \vec{a}_i, c_i\vec{a}_i \rangle$

$c_i \langle \vec{a}_i, \vec{a}_i \rangle$

$\langle \vec{a}_i, c_1\vec{a}_1 \rangle + \langle \vec{a}_i, c_2\vec{a}_2 \rangle + \cdots + \langle \vec{a}_i, c_N\vec{a}_N \rangle = 0$

$\langle \vec{a}_i, \vec{a}_j \rangle = 0 \quad i \neq j$

$c_1 \cancel{\langle \vec{a}_i, \vec{a}_1 \rangle}^{0} + c_2 \cdot 0 + \cdots + c_i \boxed{1} + c_{i+1} \cdot 0 + \cdots 0 = 0$

All c_i for $i=1,\dots,N = 0$. (A has lin indep. cols)

(2) $A\vec{x}=\vec{b}$ (A^{-1} exists because of (1)) $\vec{x}=A^{-1}\vec{b}$

can span any $\vec{b} \in \mathbb{R}^N$ because A^{-1} exists.

(c) When $\mathbf{A} \in \mathbb{R}^{N\times M}$ and $N \geq M$ (i.e. tall matrices), show that if the matrix is orthonormal, then $\mathbf{A}^T\mathbf{A} = \mathbf{I}_{M\times M}$.

$$A = \begin{bmatrix} \; \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_M \\ | & | & & | \end{bmatrix}$$

$$A^TA = \underbrace{\begin{bmatrix} — \vec{a}_1^T — \\ — \vec{a}_2^T — \\ \vdots \\ — \vec{a}_M^T — \end{bmatrix}}_{M\times N} \underbrace{\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_M \\ | & | & & | \end{bmatrix}}_{N\times M}$$

$\vec{a}_1^T\vec{a}_1$

$$(A^TA)_{ij} = \langle \vec{a}_i, \vec{a}_j \rangle$$
$$= 1 \quad i=j$$
$$= 0 \quad i\neq j$$

$$= \begin{bmatrix} \langle \vec{a}_1, \vec{a}_1 \rangle & \langle \vec{a}_1, \vec{a}_2 \rangle & \cdots \\ \vdots & & \\ & & \ddots \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{M\times M}$$

(d) Again, suppose $\mathbf{A} \in \mathbb{R}^{N\times M}$ where $N \geq M$ is an orthonormal matrix. Show that the projection of $\vec{y}$ onto the subspace spanned by the columns of $\mathbf{A}$ is now $\mathbf{A}\mathbf{A}^T\vec{y}$.

$$(A^TA)_{ij} = a_{1i}a_{1j} + a_{2i}a_{2j} + \cdots + a_{Mi}a_{Mj}$$

(b) A if orthonormal has lin indep columns

(a) A has lin indep columns $\Rightarrow$ $\text{proj}_{C(A)}\vec{y} = \underset{N\times M}{A}\,\underset{M\times M}{(A^TA)^{-1}}\,\underset{M\times N}{A^T}\vec{y}$

$$\text{proj}_{C(A)}\vec{y} = A(A^TA)^{-1}A^T\vec{y}$$
$$= A(I)^{-1}A^T\vec{y}$$
$$= AIA^T\vec{y} = \underbrace{AA^T}_{N\times N}\vec{y} = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_M \\ | & & | \end{bmatrix} \begin{bmatrix} \langle \vec{a}_1, \vec{y} \rangle \\ \langle \vec{a}_2, \vec{y} \rangle \\ \vdots \\ \langle \vec{a}_M, \vec{y} \rangle \end{bmatrix}$$

$I(I^{-1}) = I$

$I^{-1} = I$

(e) Given $\mathbf{A} \in \mathbb{R}^{N\times M} = \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and the columns of $\mathbf{A}$ are orthonormal, find the least squares solution to $\mathbf{A}\hat{\vec{x}} = \vec{y}$ where $\vec{y} = \begin{bmatrix} 5 & 12 & 7 & 8 \end{bmatrix}^T$.

$$\text{proj}_{C(A)}\vec{y} = \text{proj}_{\vec{a}_1}\vec{y} + \text{proj}_{\vec{a}_2}\vec{y} + \text{proj}_{\vec{a}_3}\vec{y}$$

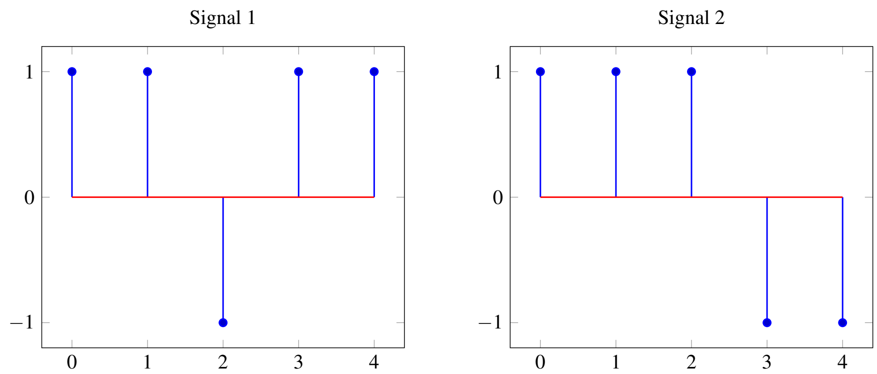
$$\text{proj}_{\vec{a}_i}\vec{y} = \frac{\langle \vec{y}, \vec{a}_i \rangle}{\langle \vec{a}_i, \vec{a}_i \rangle}\vec{a}_i$$
$$= \langle \vec{y}, \vec{a}_i \rangle \vec{a}_i$$

Normalized $\vec{a}_i$

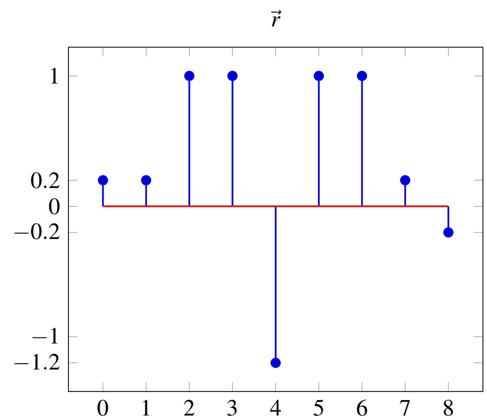
$$\hat{\vec{x}} = \underline{(A^TA)^{-1}}A^T\vec{y}$$
$$= A^T\vec{y} = \begin{bmatrix} \langle \vec{a}_1, \vec{y} \rangle \\ \langle \vec{a}_2, \vec{y} \rangle \\ \langle \vec{a}_3, \vec{y} \rangle \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 17\frac{\sqrt{2}}{2} \end{bmatrix}$$

2. Identifying satelites and their delays

We are given the following two signals, $\vec{s}_1$ and $\vec{s}_2$ respectively, that are signatures for two satellites.



(a) Your cellphone antenna receives the following signal $r[n]$. You know that there may be some noise present in $r[n]$ in addition to the transmission from the satellite.



Which satellites are transmitting? What is the delay between the satellite and your cellphone? Use cross-correlation to justify your answer. You can use iPython to compute the cross-correlation.

$$\text{corr}_{\vec{r}}(\vec{s}_1)[k]$$

(b) Now your cellphone receives a new signal $r[n]$ as below. What the satellites that are transmitting and what is the delay between each satellite and your cellphone?