EECS16A DIS 13A

Thanksgiving this week - have a good rest, no dis this wednesday!

Learning Objectives

- Geometric notion of projection of a vector onto another vector
- How to compute vector projection onto another vector
- Geometric notion of projection of a vector onto a subspace
- How to compute - (least squares formula)
- Least squares : a special case - orthogonal basis for subspace

$$\underline{A}\vec{x} \approx \vec{b}$$

Music

- Kygo - Firestone
- Devil Town - Cavetown
- Kaskade - Atmosphere

EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 13A

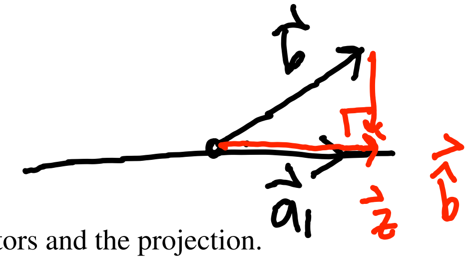
1. Mechanical Projection

In $\mathbb{R}^n$, the vector valued projection of vector $\vec{b}$ onto vector $\vec{a}$ is defined as:

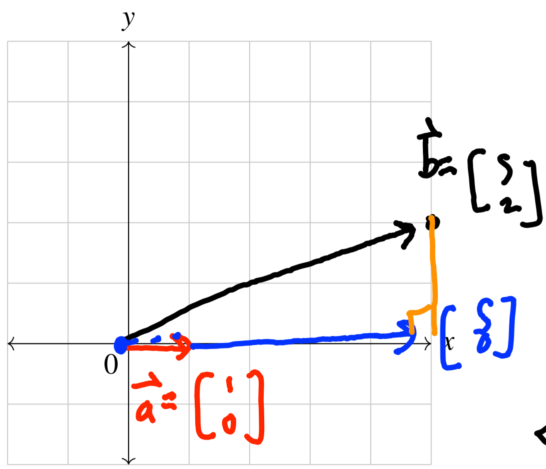
$$\text{proj}_{\vec{a}}\left(\vec{b}\right) = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\|\vec{a}\|^2}\vec{a}.$$

Recall $\|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle$.

$\vec{z} = \alpha \vec{a}_1$



(a) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ – that is, onto the x-axis. Graph these two vectors and the projection.



$$\text{proj}_{\vec{a}}(\vec{b}) = \frac{\left\langle \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

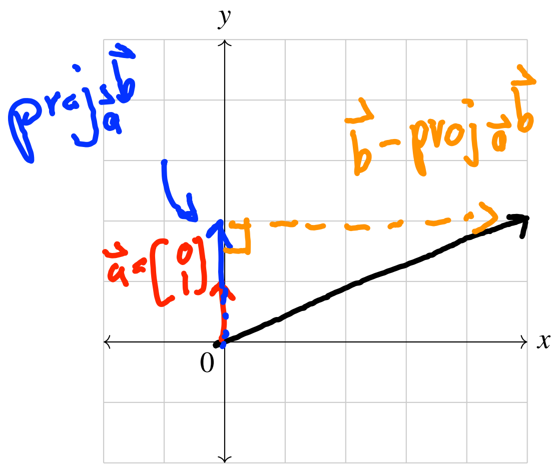
$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad = \frac{5 \cdot 1 + 2 \cdot 0}{1 \cdot 1 + 0 \cdot 0} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\langle \cdot, \cdot \rangle \qquad = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

scalar valued

(b) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ – that is, onto the y-axis. Graph these two vectors and the projection.

$\|\vec{a}\|^2 = \langle \vec{a}, \vec{a} \rangle$



$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\left\langle \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{2}{1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\langle \vec{a}, \vec{b} \rangle}{\|\vec{a}\|^2} \vec{a}$$

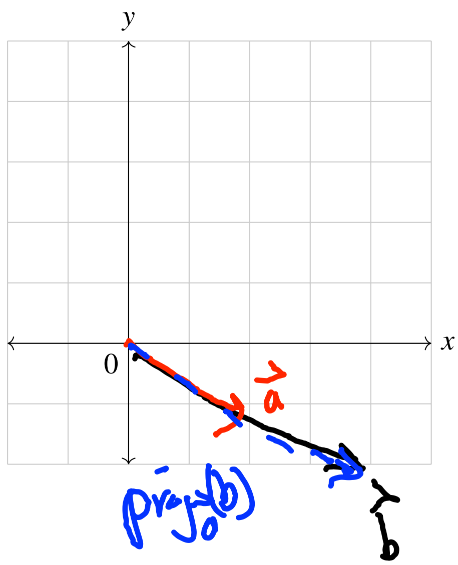
$$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

$$= \frac{\cancel{\|\vec{a}\|} \|\vec{b}\| \cos\theta}{\|\vec{a}\|^{\cancel{2}}} \vec{a}$$

$$= \underbrace{\|\vec{b}\| \cos\theta}\ \frac{\vec{a}}{\|\vec{a}\|}$$

↑ vector of length 1

(c) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Graph these two vectors and the projection.

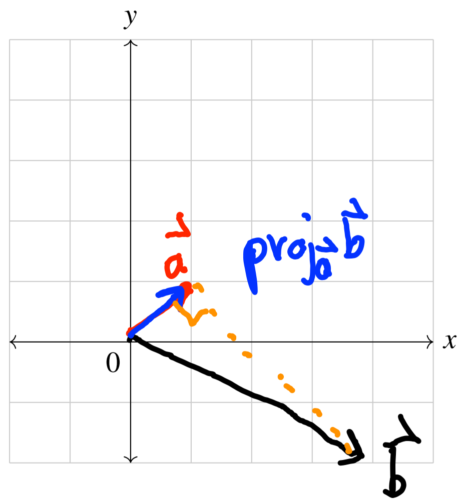


$$\mathrm{proj}_{\vec{a}}\vec{b} = \frac{\left\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\rangle} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \frac{10}{5} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

(d) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.



$$\text{proj}_{\vec{a}}(\vec{b}) = \frac{\left\langle \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

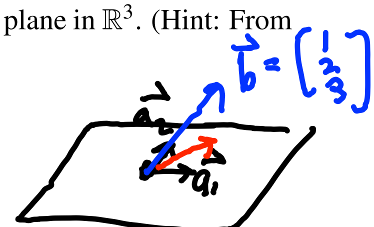
$$= \frac{2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Projection doesn't change if we change length of $\vec{a}$. $\vec{a}_1$ $\vec{a}_2$

$\vec{b} =$

(e) Project $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the span of the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ – that is, onto the x-y plane in $\mathbb{R}^3$. (Hint: From least squares, the matrix $A(A^\top A)^{-1}A^\top$ projects a vector into $C(A)$.)



$\vec{z} = A\vec{x}$ $\quad \hat{\vec{x}} = (A^TA)^{-1}A^T\vec{b}$ $\quad A\vec{x} \approx \vec{b}$

$\vec{b}$

$$\text{proj}_{C(A)}(\vec{b}) = A(A^TA)^{-1}A^T\vec{b}$$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

(f) What is the geometric/physical interpretation of projection? Justify using the previous parts.

Minimize distance between vector we approximate and our approximation

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Bigg(\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{2\times 3} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{3\times 2} \Bigg)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}}_{2\times 2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

* look @ ANS for visualization

2. Least Squares with Orthogonal Columns

Suppose we would like to solve the least squares problem for $\mathbf{A} \in \mathbb{R}^{3\times 2}$ and $\vec{b} \in \mathbb{R}^3$; that is, find an optimal vector $\vec{x} \in \mathbb{R}^2$ which gets $\mathbf{A}\vec{x}$ closest to $\vec{b}$ such that the distance $||\vec{e}|| = ||\vec{b} - \mathbf{A}\vec{x}||$ is minimized. Call this optimal vector $\vec{\hat{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$. Mathematically, we can express this as:

$$||\vec{b} - \mathbf{A}\vec{\hat{x}}||^2 = \min_{\vec{x}\in\mathbb{R}^2} ||\vec{b} - \mathbf{A}\vec{x}||^2 = \min_{\vec{x}\in\mathbb{R}^2} \left\| \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2.$$

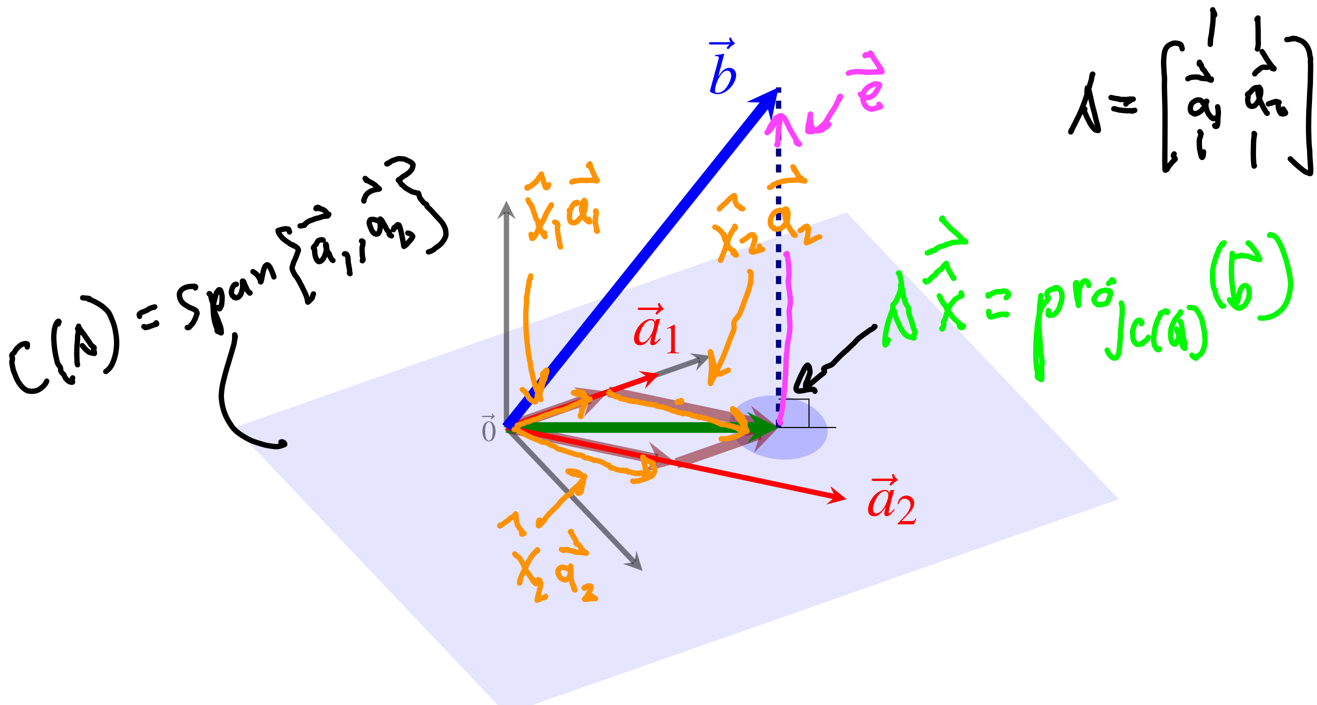
To identify the solution $\vec{\hat{x}}$, we may recall the least squares formula: $\vec{\hat{x}} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\vec{b}$, which is applicable when $\mathbf{A}$ has linearly independent columns. We would now like to walk through the intuition behind this formula for the case when $\mathbf{A}$ has orthogonal columns: $\langle\vec{a}_1, \vec{a}_2\rangle = 0$.

(a) On the diagram below, please label the following elements:

$A\vec{\hat{x}} = \underline{\hat{x}_1}\vec{a}_1 + \underline{\hat{x}_2}\vec{a}_2$

NOTE: For this sub-part only, the matrix $\mathbf{A}$ does not have orthogonal columns.

span $\{\vec{a}_1, \vec{a}_2\}$ $\quad$ $\mathbf{A}\,\vec{\hat{x}}$ $\quad$ $\hat{x}_1\,\vec{a}_1$ $\quad$ $\hat{x}_2\,\vec{a}_2$ $\quad$ $C(\mathbf{A})$ $\quad$ $\vec{e} = \vec{b} - \mathbf{A}\,\vec{\hat{x}}$ $\quad$ $\text{proj}_{C(\mathbf{A})}(\vec{b})$.



* Look @ ANS for prettier picture

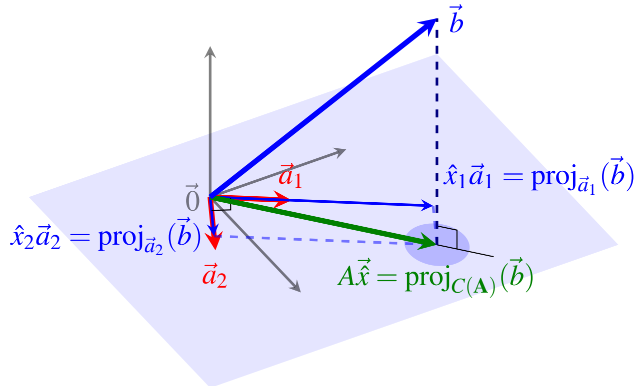
(b) Now suppose we assume a special case of the least squares problem where the columns of $\mathbf{A}$ are orthogonal (illustrated in the figure below). Given that $\vec{\hat{x}} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\vec{b}$, and $\text{proj}_{C(\mathbf{A})}(\vec{b}) = \mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\vec{b} =$

$\mathbf{A}\vec{\hat{x}}$, show the following statement holds.

$$\langle \vec{a}_1, \vec{a}_2 \rangle = 0 \quad \implies \quad \vec{\hat{x}} = \begin{bmatrix} \frac{\langle \vec{a}_1, \vec{b} \rangle}{||\vec{a}_1||^2} \\ \frac{\langle \vec{a}_2, \vec{b} \rangle}{||\vec{a}_2||^2} \end{bmatrix} \quad \text{and} \quad \text{proj}_{C(\mathbf{A})}(\vec{b}) = \text{proj}_{\vec{a}_1}(\vec{b}) + \text{proj}_{\vec{a}_2}(\vec{b})$$

In words, the statement says that when the columns of $\mathbf{A}$ are orthogonal, the entries of the least squares solution vector $\vec{\hat{x}}$ can be computed by using $\vec{b}$ and only the single other vector $\vec{a}_i$, and that the projection of $\vec{b}$ onto $C(\mathbf{A})$ can be computed by summing the projections of $\vec{b}$ onto the $\vec{a}_i$.

RECALL... $\quad \text{proj}_{\vec{a}_1}(\vec{b}) = \frac{\langle \vec{a}_1, \vec{b} \rangle}{||\vec{a}_1||^2} \vec{a}_1, \quad ||\vec{a}||^2 = \langle \vec{a}, \vec{a} \rangle \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}$



(c) Compute the least squares solution $\vec{\hat{x}} \in \mathbb{R}^2$ to the following system:

$$\min_{\vec{x} \in \mathbb{R}^2} \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2.$$

HINT: Notice that the columns of $\mathbf{A}$ are orthogonal!!

$$\hat{x}_1 = \frac{\vec{a}_1^T \vec{b}}{\vec{a}_1^T \vec{a}_1} = \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \frac{1}{1}$$

$$\hat{x}_2 = \frac{\vec{a}_2^T \vec{b}}{\vec{a}_2^T \vec{a}_2} = \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} = \frac{2}{1}$$

$$\boxed{\vec{\hat{x}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$\hat{\vec{x}}$, $A\hat{\vec{x}} = \text{proj}_{C(A)}\vec{b} = A(A^TA)^{-1}A^T\vec{b}$

WTS: $\text{proj}_{C(A)}\vec{b} = \text{proj}_{\vec{a}_1}\vec{b} + \text{proj}_{\vec{a}_2}\vec{b}$

$$\hat{\vec{x}} = (A^TA)^{-1}A^T\vec{b} \qquad A = \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}$$

$$= \left(\begin{bmatrix} - \vec{a}_1^T - \\ - \vec{a}_2^T - \end{bmatrix}\begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix}\right)^{-1}\begin{bmatrix} - \vec{a}_1^T - \\ - \vec{a}_2^T - \end{bmatrix}\vec{b}$$

$\vec{a}_1^T\vec{a}_2 =$
$\vec{a}_2^T\vec{a}_1 = 0$

$$= \begin{bmatrix} \vec{a}_1^T\vec{a}_1 & 0 \\ 0 & \vec{a}_2^T\vec{a}_2 \end{bmatrix}^{-1}\begin{bmatrix} \vec{a}_1^T\vec{b} \\ \vec{a}_2^T\vec{b} \end{bmatrix}$$

$$\begin{bmatrix} \|\vec{a}_1\|^2 & 0 \\ 0 & \|\vec{a}_2\|^2 \end{bmatrix} S = I$$

$$= \begin{bmatrix} \|\vec{a}_1\|^2 & 0 \\ 0 & \|\vec{a}_2\|^2 \end{bmatrix}^{-1}\begin{bmatrix} \vec{a}_1^T\vec{b} \\ \vec{a}_2^T\vec{b} \end{bmatrix}$$

$$S = \begin{bmatrix} \frac{1}{\|\vec{a}_1\|^2} & 0 \\ 0 & \frac{1}{\|\vec{a}_2\|^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\|\vec{a}_1\|^2} & 0 \\ 0 & \frac{1}{\|\vec{a}_2\|^2} \end{bmatrix}\begin{bmatrix} \vec{a}_1^T\vec{b} \\ \vec{a}_2^T\vec{b} \end{bmatrix} = \begin{bmatrix} \frac{\vec{a}_1^T\vec{b}}{\|\vec{a}_1\|^2} \\ \frac{\vec{a}_2^T\vec{b}}{\|\vec{a}_2\|^2} \end{bmatrix}$$

$$\text{proj}_{C(A)}\vec{b} = A\hat{\vec{x}} = \frac{\vec{a}_1^T\vec{b}}{\|\vec{a}_1\|^2}\vec{a}_1 + \frac{\vec{a}_2^T\vec{b}}{\|\vec{a}_2\|^2}\vec{a}_2$$

$$= \text{proj}_{\vec{a}_1}\vec{b} + \text{proj}_{\vec{a}_2}\vec{b} \in$$

Q: $A = \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix}$ $\vec{a}_i \in \mathbb{R}^3$

$\text{proj}_{C(A)}\vec{b} = ?$ $(\vec{b} \in \mathbb{R}^3)$

A: $= \vec{b}$

Q : What happens when $\langle \vec{a}_1, \vec{a}_2 \rangle \neq 0$

$$\hat{\vec{x}} = \begin{bmatrix} \vec{a}_1^T \vec{a}_1 & \vec{a}_1^T \vec{a}_2 \\ \vec{a}_2^T \vec{a}_1 & \vec{a}_2^T \vec{a}_2 \end{bmatrix}^{-1} \begin{bmatrix} \vec{a}_1^T \vec{b} \\ \vec{a}_2^T \vec{b} \end{bmatrix}$$

$$= \frac{1}{\|\vec{a}_1\|^2 \|\vec{a}_2\|^2 - 2\,\vec{a}_1^T \vec{a}_2} \begin{bmatrix} \|\vec{a}_2\|^2 & -\vec{a}_1^T \vec{a}_2 \\ -\vec{a}_1^T \vec{a}_2 & \|\vec{a}_1\|^2 \end{bmatrix} \begin{bmatrix} \vec{a}_1^T \vec{b} \\ \vec{a}_2^T \vec{b} \end{bmatrix}$$

$$\neq \text{proj}_{\vec{a}_1} \vec{b} + \text{proj}_{\vec{a}_2} \vec{b}$$