EECS16A DIS6A

MT1 is tonight! 7-9PM PST

Review logistics on piazza posts @525, @587

A few choices for today's discussion

Worksheet topics

	Votes
① Matrix transformations - finding them	16
② Matrix transformations - Rotations/Scaling	11
③ Gaussian Elimination	3
④ Eigenvectors/Eigenspaces	30 ← Second
⑤ Nullspace/Invertibility/Proof	50 ← First
→ Questions on all past discussion material (of your choosing!)	20 ← third

(New thing I'm trying out)

→ Music before Berkeley time

Two tracks:

- Natalia Lafourcade - Soledad y El Mar
- Raveena - Honey

→ Suggest tracks @ bit.ly/16ajukebox

trivial nullspace $\{\vec{0}\}$

(nullspace of A) Set of all vectors $\vec{v}$ that make $A\vec{v} \to \vec{0}$

$A\vec{0} = \vec{0}$ always true no matter what

2. (Optional) Proof

Concept: Null Spaces, Invertibility

Consider a square matrix **A**. Prove that if **A** has a non-trivial nullspace, i.e. if the nullspace of **A** contains more than just $\vec{0}$, then matrix **A** is not invertible.

What do we know: $\underset{n \times n}{A}$, $A\vec{v} = \vec{0}$ there are $\vec{v} \neq \vec{0}$ that the above eqn true.

What to show: A^{-1} doesn't exist.

Attempt by contradiction

Assume that A^{-1} does exist (hoping for contradiction)

$A\vec{v} = \vec{0}$

$A^{-1}A\vec{v} = A^{-1}\vec{0}$

$I\vec{v} = \vec{0}$

$\vec{v} = \vec{0}$ → The only solution to $A\vec{v} = \vec{0}$ if the inverse exists must be $\vec{v} = \vec{0}$.

But our given/what we know says that there are $\vec{v} \neq \vec{0}$ that satisfy $A\vec{v} = \vec{0}$

$\vec{v} = \vec{0}$ and $\vec{v} \neq \vec{0}$ is a contradiction

Therefore A cannot be invertible

Q: Why can we assume $\vec{v} \neq \vec{0}$?

A: Because we were told that A has nontrivial nullspace

Q: Strategy for proof type - which to choose?

A: Direct proof → can do computations/do math by using definitions or givens

Constructive proof → know what solution should look like and can write expressions for which you can guarantee properties

Contradiction → Don't see easy method for manipulation
→ the want to show part seems 'interesting'

(d) *Concept: Eigenspaces/Eigenvectors/Eigenvalues*

It turns out the Romulan engineers were not as smart the Enterprise engineers. Their calculations did not work out and they positioned the probe at (x_q, y_q) such that the *cloaking* (transformation) matrix, $\mathbf{A}_p$, mapped it to (u_q, v_q), where

$$\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \mathbf{A}_p \begin{bmatrix} x_q \\ y_q \end{bmatrix}, \text{ and } \mathbf{A}_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

As a result, the torpedo while traveling along a straight line from $(0,0)$ to (u_q, v_q), hit the probe at (x_q, y_q) on the way!

The scenario is shown in Figure 4. For the torpedo to hit the probe, we must have $\begin{bmatrix} u_q \\ v_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$, where λ is a real number.

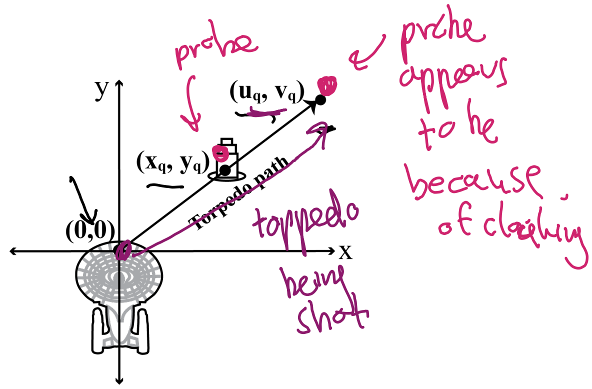

Figure 4: Figure for part (d)

Find the possible positions of the probe (x_q, y_q) so that $(u_q, v_q) = (\lambda x_q, \lambda y_q)$. Remember that the torpedo cannot be fired on its initial position $(0,0)$.

$\begin{bmatrix} u_q \\ v_q \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} u_q \\ v_q \end{bmatrix} = A_p \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} \lambda x_q \\ \lambda y_q \end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - \lambda I\right) = \det\left(\begin{bmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{bmatrix}\right) = (1-\lambda)(6-\lambda) - 6$$
$$= \lambda^2 - 7\lambda + 6 - 6$$
$$= \lambda^2 - 7\lambda \rightarrow \boxed{\lambda_1 = 0} \leftarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \lambda_2 = 7$$

means probe appears to be ⓐ

Have to use $\lambda_2 = 7$

$$\begin{bmatrix} -6 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_1/_{-6} \to R_1} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$(A_p - \lambda_2 I)\vec{x} = \vec{0}$

$$\xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad x_q - \tfrac{1}{2} y_q = 0, \quad x_q = \tfrac{1}{2} y_q$$

$x_q = \frac{1}{2} t$

$y_q = t$

$$\rightarrow \text{Span}\left\{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right\} \qquad \boxed{\begin{bmatrix} x_q \\ y_q \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} t, \ t \neq 0}$$

want to exclude $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ 7 \end{bmatrix}$$

Q: Why are eigenvalues / eig. vec. important in imaging?

A: $\vec{m} = M \vec{i}$

$\vec{m}$ measurements, M masking, $\vec{i}$ image

$\vec{m} = M\vec{i} + \vec{w}$ ← noise

M invertible

Assume that $\vec{w}$ is a linear combination of M's eigenvectors

$\vec{w} = \alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$

$M\vec{w} = \lambda_1\alpha_1\vec{v}_1 + \cdots \lambda_n\alpha_n\vec{v}_n$

$M^{-1}\vec{v}_i = \frac{1}{\lambda_i}\vec{v}_i$

$\vec{v}_i$ eigvec of M, $\frac{1}{\lambda_i}$ eigenval of M^{-1}

$\hat{i} = M^{-1}\vec{m} = M^{-1}(M\vec{i} + \vec{w})$

$\hat{i}$ estimate of image

$= \vec{i} + \underline{M^{-1}\vec{w}}$

Q: Explain note 9.8.2 - representing $\vec{x}[0]$ as linear comb. of eigenvectors

A:

$$\boxed{M^{-1}\vec{w} = \underline{\frac{1}{\lambda_1}\alpha_1\vec{v}_1 + \cdots + \frac{1}{\lambda_n}\alpha_n\vec{v}_n}}$$

⇒ shrinks if λ are large

$\vec{x}[0] = \alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$ → eigvecs same transition matrix T

$\vec{x}[0]$ ← state

$\vec{x}[n+1] = T\vec{x}[n]$

$\vec{x}[1] = T\vec{x}[0] = \lambda_1\alpha_1\vec{v}_1 + \cdots + \lambda_n\alpha_n\vec{v}_n$

$\vec{x}[2] = T\vec{x}[1] = \lambda_1^2\alpha_1\vec{v}_1 + \cdots + \lambda_n^2\alpha_n\vec{v}_n$

$\vec{x}[n] = T^m\vec{x}[0] = \lambda_1^m\alpha_1\vec{v}_1 + \cdots + \lambda_n^m\alpha_n\vec{v}_n$

can express $\vec{x}[0]$ as lin. comb. of eigenvectors only if all eigenvalues have enough eigenvectors

1. all eigenvalues distinct
2. all eigenspaces → sum of dim = num of rows/cols of T

lets say we have eigvecs $\vec{v}_1, \dots, \vec{v}_n$

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \vec{x}[0] \rightarrow \text{GE}$$

$$\underset{\vec{x}[1]}{T\vec{x}[0]} \longrightarrow \underset{\vec{x}[m]}{T^m \vec{x}[0]}$$