EECS16A DIS 5B

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OH: W 10AM-12PM PST (HWP)

Exam coming up!
Read piazza post with exam logistics

Learning Objectives

1. Eigenvalues of transition matrices and state behavior
   a. When are states blowing up?
   b. Shrinking? Decaying?
   c. Staying the same?
2. Eigenvalues and eigenvectors of special matrices and linear transformations
   a. Geometric transformations
   b. Nullspaces and eigenvectors

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1 a ii $(M-\lambda_2 I)\vec{v}_2 = \vec{0}$ $\quad \lambda_2 = 2$

$$\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} - 2I\right)\vec{v}_2 = \vec{0}$$

$$\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\vec{v}_2 = \vec{0} \qquad \vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

GE

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\vec{v}_2 = \vec{0}$$

$\frac{3}{2} \quad -1 - \frac{1}{2} = 0$

$$\boxed{\text{Span}\left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}}$$

1 a iii $(M-\lambda_3 I)\vec{v}_3 = \vec{0}$ $\quad \lambda_3 = \frac{1}{2}$

$$\begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -2 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}\vec{v}_3 = \vec{0} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

GE

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\vec{v}_3 = \vec{0} \qquad \boxed{\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}}$$

Q: Is span$\{\vec{v}\} = \{\vec{v}\cdot t \mid t \in \mathbb{R}\}$ A: Yes

$A\vec{0} = \lambda\vec{0}$ Q: What λ's make this true?
A: any λ that is a real # e.g. $\lambda = 5$, $\lambda = -2$

EECS 16A Fall 2020 — Designing Information Devices and Systems I — Discussion 5B

1. Steady and Unsteady States

Usually find λs $\det(A-\lambda I)=0$

(a) You're given the matrix **M**:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Which generates the next state of a physical system from its previous state: $\boxed{\vec{x}[k+1] = \mathbf{M}\vec{x}[k]}$. ($\vec{x}$ could describe either people or water.) Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$

ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$

iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

Here we have a 3x3
→ Don't expect calculation of determinant of 3x3 matrices

Q: How to find eigenvector $\vec{v}_1$, corresponding to eigenvalue λ_1?

A: (a) $N(M-\lambda_1 I) = N(M-I)$ ($\lambda_1 = 1$ substitute)

(b) Solve $(A-\lambda_1 I)\vec{v}_1 = \vec{0}$
$(A-I)\vec{v}_1 = \vec{0}$

Both correct

$M\vec{v}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\text{Eigenspace for } \lambda_1 = 1 \Rightarrow \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}}$

$M-I = \begin{bmatrix} -1/2 & 1/2 & -1/2 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} -1/2 & 1/2 & -1/2 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \overset{GE}{\Rightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \leftarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$$

converges. If it does, what does it converge to?

Q: How to write solutions for 1(a)(i)

$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{aligned} x_1 - x_2 + 0 &= 0 \\ x_3 &= 0 \\ 0 &= 0 \end{aligned}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix}$

$\lambda_1 = 1$

why $|\lambda|>1$ ←

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n \vec{x}$
0	0	$\neq 0$	Yes	$\vec{0}$
0	$\neq 0$	0	No	
0	$\neq 0$	$\neq 0$	No	
$\neq 0$	0	0	Yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	Yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	No	
$\neq 0$	$\neq 0$	$\neq 0$	No	

$M\vec{v}_1 = \lambda_1\vec{v}_1 = \vec{v}_1$

$\lambda_2 = 2$

$M\vec{v}_2 = \lambda_2\vec{v}_2 = 2\vec{v}_2$

$\lambda_3 = \frac{1}{2}$

$M\vec{v}_3 = \lambda_3\vec{v}_3 = \frac{1}{2}\vec{v}_3$

$\vec{x}[0] = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$

$M\vec{x}[0] = M(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3)$

$= \alpha M\vec{v}_1 + \beta M\vec{v}_2 + \gamma M\vec{v}_3$

$= \alpha\vec{v}_1 + \beta\, 2\vec{v}_2 + \gamma\frac{1}{2}\vec{v}_3$

$M(M\vec{x}[0]) = M^2\vec{x}[0] = \alpha\vec{v}_1 + \beta(2)^2\vec{v}_2 + \gamma\left(\frac{1}{2}\right)^2\vec{v}_3$

$\alpha(1)^2\vec{v}_1$

$M^n\vec{x}[0] = \alpha\vec{v}_1 + \beta\, 2^n\vec{v}_2 + \gamma\left(\frac{1}{2}\right)^n \vec{v}_3$

λ^n

$\lambda = 1 \Rightarrow \lambda^n = 1$

e.g. $\lambda = 2$, $\lambda = -2$

$|\lambda| > 1 \Rightarrow \lambda^n \to \infty$

e.g. $\lambda = \frac{1}{3}$, $\lambda = -\frac{1}{2}$

$|\lambda| < 1 \Rightarrow \lambda^n \to 0$

Q: What if $2^n\begin{bmatrix}1\\1\end{bmatrix} + (-3)^n\begin{bmatrix}1\\0\end{bmatrix}$ ← $n\to\infty$ doesn't converge

Q: $\lambda = -1$
↳ Doesn't converge
→ flip flop

2. Eigenvalues and Special Matrices – Visualization

As seen earlier, an eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

all λs $= 1$

all of $\mathbb{R}^n$ → any vector in $\mathbb{R}^n$ is an eigenvector

D

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

$D\vec{v} = \lambda\vec{v}$

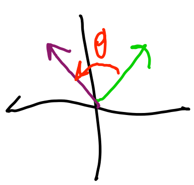
Candidate: $\lambda_1 = d_1, \lambda_2 = d_2, \ldots, \lambda_n = d_n$

Eigenvectors candidates: $\vec{v}_1 = \left.\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} n$ $\quad D\vec{v}_1 = \begin{bmatrix} d_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 \underline{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$ ← eigenvector for $\lambda_1 = d_1$

$\vec{v}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← zeroes everywhere else; ← ith row

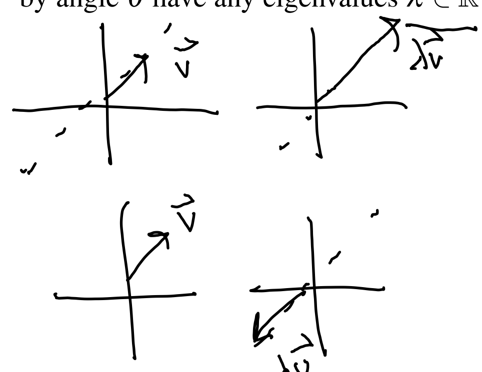
→ eigenvector for d_i

(c) Conceptually, does a rotation matrix in $\mathbb{R}^2$ by angle θ have any eigenvalues $\lambda \in \mathbb{R}$? For which angles is this the case?



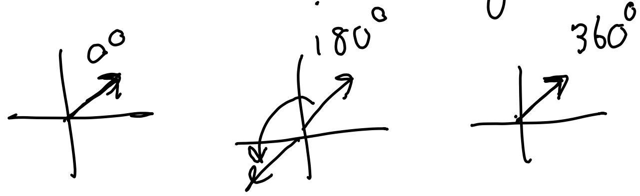
$A\vec{v} = \lambda\vec{v}$

If λ is real eigenvector is a vector that stays on the same line we started from



Rotations of $\underline{180^\circ}$ or $\underline{0^\circ}$ → keep us on same line!

Any other rotation → imaginary/complex eigenvalues



(d) Now let us mechanically compute the eigenvalues of the rotation matrix in $\mathbb{R}^2$. Does it agree with our findings above? As a refresher, the rotation matrix $\mathbf{R}$ has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\det(R - \lambda I) = \det\left(\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}\right) = (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$(\cos\theta - \lambda)^2 = -\sin^2\theta$$

$$\cos\theta - \lambda = \pm\sqrt{-\sin^2\theta}$$

$$\lambda = \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$= \cos\theta \pm \underline{i \sin\theta}$$

$$\theta = 0^\circ, 180^\circ, 360^\circ$$

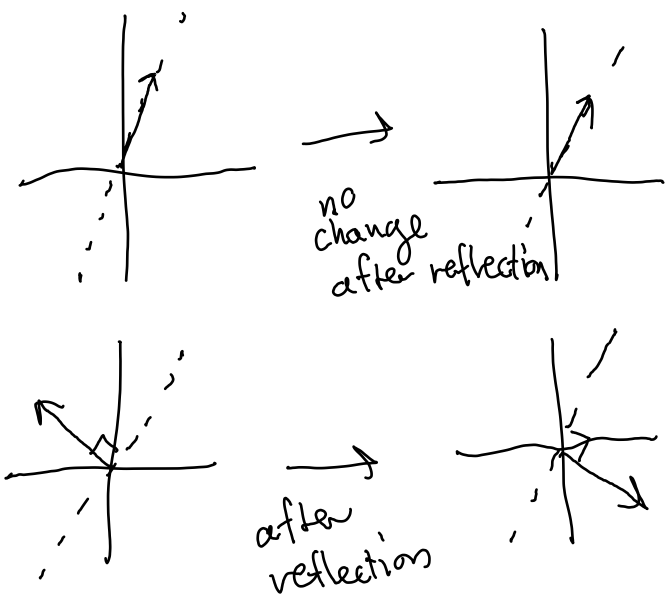
$$i\sin\theta = 0$$

$x \overset{\text{in}}{\in} \mathbb{R} \leftarrow$ set of number

" x is a real " number

(e) Does the reflection matrix $\mathbf{T}$ across the x-axis in $\mathbb{R}^{2\times 2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$\lambda = 1$

$\lambda = -1$

All reflection matrices have

$$\begin{cases} \lambda = 1 \\ \lambda = -1 \end{cases}$$

(f) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

Q: No solution case example?

A: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

no solution

$\lambda = 0$

$\lambda = 1$

$M\vec{v} = \lambda\vec{v} \quad \lambda = 0$

$M\vec{v} = \vec{0} \quad (\vec{v} \neq \vec{0})$

$\Rightarrow$ M has nontrivial nullspace.

$\Rightarrow$ Many or no solutions

$M\vec{v} = \vec{0}$ another solution if $\vec{x}$ is a solution

$M(\vec{x} + \vec{v}) \stackrel{?}{=} \vec{b}$

$M\vec{x} + M\vec{v} \stackrel{?}{=} \vec{b}$

$M\vec{x} + \vec{0} \stackrel{\checkmark}{=} b$

(g) **(Practice)** Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?