EECS16A DIS 5A

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OH: W 10AM-12PM PST (HW Party)

Last section (4B) was a bit fast in my opinion.
If you feel this way and there's some feedback you have, fill out bit.ly/mosesfb (anonymous feedback form)

Learning Objectives

Definition
An eigenvector $\vec{v} \neq \vec{0}$ of a matrix A with eigenvalue λ satisfies
$$A\vec{v} = \lambda\vec{v}$$
Intuition: doesn't change direction

$\boxed{1}$ Compute eigenvalues, eigenvectors, eigenspaces of matrix A

Procedure

1. Compute $f(\lambda) = \det(A - \lambda I)$ (aka characteristic polynomial of A)
2. Compute eigenvalues, λ_i, which make $f(\lambda_i) = 0$
3. For each eigenvalue, λ_i, find eigenvectors / eigenspace
   - (i) Compute $A - \lambda_i I$
   - (ii) Solve for $\vec{v}_i$, eigenvectors where $(A - \lambda_i I)\vec{v}_i = \vec{0}$ (i.e. find nullspace of $A - \lambda_i I$ / $N(A - \lambda_i I)$)

$\boxed{2}$ Definition of steady state + How to find it (eigenvectors of $\lambda = 1$)

$\boxed{3}$ Relationship between eigenvalues and invertibility
$\Rightarrow$ The nullspace is an eigenspace.

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Recall from lecture the way to compute a determinant of any 2×2 matrix is by using the following formula:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(\mathbf{A}) = ad - bc$$

1. Mechanical Eigenvalues and Eigenvectors

Definition: For some matrix $\mathbf{A}$, the polynomial function of λ, $f(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I})$, is known as the *characteristic polynomial* of $\mathbf{A}$.

Find the eigenvalues (which are the roots of the characteristic polynomial) of each matrix $\mathbf{M}$ and their associated eigenvectors. State if the inverse of $\mathbf{M}$ exists.

(a) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

① Compute $\det(M - \lambda I)$

$$\det\left(\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right)$$

$$= -\lambda(-3-\lambda) - (-2)(1)$$

$$f(\lambda) = \lambda^2 + 3\lambda + 2$$

② Finding λ_i that make $f(\lambda_i) = 0$

$f(\lambda) = 0$?

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 2)(\lambda + 1) = 0$

$\lambda + 2 = 0$ or $\lambda + 1 = 0$

$\lambda_1 = -2$ $\lambda_2 = -1$

③ For each λ_i, find $M - \lambda_i I$ and $N(M - \lambda_i I)$

$\lambda_1 = -2$ $M - \lambda_1 I = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \overset{GE}{\Rightarrow} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 + \frac{1}{2}t = 0$, $x_2 = t$ $\Rightarrow \vec{v}_1 = \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix}$

$M\vec{v}_1 = \lambda_1 \vec{v}_1 \leftarrow (M - \lambda_1 I)\vec{v}_1 = \vec{0}$

$M\vec{v}_1 - \lambda_1 \vec{v}_1 = \vec{0}$

$M\vec{v}_1 = \lambda_1 \vec{v}_1$

$\boxed{\vec{v}_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}}$ eigenvector

$\boxed{\text{eigenspace} \quad \text{Span}\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right\}}$

(b) $\mathbf{M} = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$

$\det(M-\lambda I) = f(\lambda)$

(c) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2 - 0(1) = \lambda^2$ $\lambda_1 = 0$, $\lambda_2 = 0$

$f(\lambda) = \lambda^2 = 0$

$\lambda_2 = \lambda_1 = 0$ $(M-\lambda I)\vec{v} = \vec{0}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{v} = \vec{0} \rightarrow \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\vec{v} = \begin{bmatrix} t \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$ $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -5 \\ 0 \end{bmatrix} = \vec{0} = 0 \cdot \begin{bmatrix} -5 \\ 0 \end{bmatrix}$ ✓

$\det(M-\lambda I)$

(d) $\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. $\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - (-1)(1)$

$f(\lambda) = \lambda^2 + 1 = 0$ $\begin{cases} \lambda_1 = i \\ \lambda_2 = -i \end{cases}$

$\lambda^2 = -1$

$\lambda = \pm\underbrace{\sqrt{-1}}_{i} = \pm i$

$(M-\lambda_1 I)\vec{v}_1 = \vec{0}$

$\lambda_1 = i$ $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\vec{v}_1 = \vec{0}$

$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{GE} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_2 = -i$ $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\vec{v}_2 = \vec{0}$

$(M-\lambda_2 I)\vec{v}_2 = \vec{0}$

$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{GE} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

2. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

$*$ → conjugate

$(a+bi)^* = a - bi$ $a^* = a$

$M\vec{v} = \lambda\vec{v}$

$(M\vec{v})^* = (\lambda\vec{v})^*$

$M^*\vec{v}^* = \lambda^*\vec{v}^*$

$M\vec{v}^* = \lambda^*\vec{v}^*$

③ continued

[I] ⓐ $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$\underline{\lambda_2 = -1}$ $M - \lambda_2 I = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$

$(M - \lambda_2 I)\vec{v}_2 = \vec{0}$

eigenvector

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \overset{GE}{\Rightarrow} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 + t = 0$ $x_1 = -t$, $x_2 = t$

$\lambda_2 = -1$	$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$
eigenvalue	eigenvector	eigenspace

$M\vec{v} = \lambda\vec{v}$

$\lambda_1 = -2$

$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \underline{\begin{bmatrix} 1 \\ -2 \end{bmatrix} \overset{?}{=} -2\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}$

$\lambda_2 = -1$ $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \overset{?}{=} (-1)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ✓

[I] ⓑ $M = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$ $\det(M - \lambda I)$

If nullspace trivial: every eigenvalue is not 0

$\det\left(\begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$\rightarrow \det\left(\begin{bmatrix} -2-\lambda & 4 \\ -4 & 8-\lambda \end{bmatrix}\right) = (-2-\lambda)(8-\lambda) - (4)(-4)$

$= \lambda^2 - 6\lambda \cancel{-16} \cancel{+16}$

$= \lambda(\lambda - 6) = 0$

$\lambda_1 = 0$

$\lambda_2 = 6$

$\lambda_1 = 0$ $\overset{M - \lambda I}{\begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}}\vec{v}_1 = \vec{0} \Rightarrow \left[\begin{array}{cc|c} -2 & 4 & 0 \\ -4 & 8 & 0 \end{array}\right] \overset{GE}{\Rightarrow} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \leftarrow$ $x_2 = t$, $\underline{x_1 = 2t}$

$N(M - \lambda I)$

$N(M)$

eigenvector

eigenspace $\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$\lambda_2 = 6$ $\overset{M - 6I}{\begin{bmatrix} -2-6 & 4 \\ -4 & 8-6 \end{bmatrix}} = \begin{bmatrix} -8 & 4 \\ -4 & 2 \end{bmatrix}$

$\begin{bmatrix} -8 & 4 \\ -4 & 2 \end{bmatrix}\vec{v}_2 = \vec{0}$

$\left[\begin{array}{cc|c} -8 & 4 & 0 \\ -4 & 2 & 0 \end{array}\right] \overset{GE}{\Rightarrow} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\vec{v}_2 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ could also do $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

eigenvector

eigenspace $\text{span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$

$A^{-1} = \frac{1}{\underline{ad-bc}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ A is invertible if $\det(A) \neq 0$

$\det(A - \underset{\lambda=0}{\lambda} I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)$

$\underline{\det(A)} = \lambda_1 \cdot \lambda_2$

$\lambda_i = 0 \Rightarrow$ nontrivial nullspace

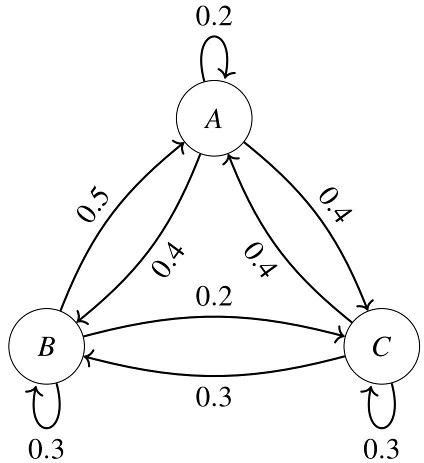

Figure 1: Reservoir pumps system.

Q: Why $N(M-\lambda I)$?

A: $\vec{v} \in N(M-\lambda I)$

$(M-\lambda I)\vec{v} = \vec{0}$

$\Rightarrow M\vec{v} - \lambda I \vec{v} = \vec{0}$

$\rightarrow \underline{M\vec{v} = \lambda \vec{v}}$

$\vec{v} \neq \vec{0}$

(a) Write out the transition matrix **T** representing the pumps system.

$$x_A[n+1] = 0.2\,x_A[n] + 0.5\,x_B[n] + 0.4\,x_C[n]$$
$$x_B[n+1] = 0.4\,x_A[n] + 0.3\,x_B[n] + 0.3\,x_C[n]$$
$$x_C[n+1] = 0.4\,x_A[n] + 0.2\,x_B[n] + 0.3\,x_C[n]$$

$$T = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of **T**. Find a steady state vector $\vec{x}$, i.e. a vector such that $\underline{T\vec{x} = \vec{x}}$.

Defn: Steady state vector $\rightarrow T\vec{x} = \vec{x}$

$\Updownarrow$ $\quad T\vec{x} = 1 \cdot \vec{x} \quad \lambda = 1$

Eigenvector of eigenvalue 1

$(T-I)\vec{x} = \vec{0}$

$(T-\lambda I)\vec{x} = \vec{0}$ $\quad \rightarrow T-I$

$$\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix} \vec{x} = \vec{0}$$

$$\left[\begin{array}{c|c} T-I & \vec{0} \end{array}\right] \xrightarrow{GE} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{43}{36} & 0 \\ 0 & 1 & -\frac{10}{9} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \vec{x} = \begin{bmatrix} \frac{43}{36} \\ \frac{10}{9} \\ 1 \end{bmatrix}$$

$\uparrow$ solutions to this are in $N(T-I)$

Q: $(A-\lambda I)\vec{v}=\vec{0}$

is this possible

$$\cancel{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\vec{v}=\vec{0}}$$

$\hookrightarrow \vec{v}=\vec{0}$

A: No, doesn't happen.

Never get $\vec{0}$ as an eigenvector

but $\vec{0}$ is in eigenspaces

(we typically don't call $\vec{0}$ eigenvector)

$\underline{A\vec{0}=\lambda\vec{0}} \rightarrow$ not useful

Q: How many eigenvalues for a matrix ?

(square matrix)

A: Always n where $\underset{n\times n}{A}$

Q: What is the relationship of trivial nullspace to eigenvalues?

A: A has trivial nullspace

$\Leftrightarrow$ The vector $\vec{x}$, $A\vec{x}=\vec{0}$ is $\vec{x}=\vec{0}$

① When we have all eigenvalues nonzero $\longrightarrow \det(A)\neq 0$

$\Rightarrow$ trivial nullspace

$\Rightarrow$ A is invertible

② When we have at least one zero eigenvalue

$A\vec{v}=0\cdot\vec{v}$ has solution $\vec{v}\neq\vec{0}$

A has nontrivial nullspace

$N(A)\neq\{\vec{0}\}$

$\uparrow$ will have more stuff