EECS16A DIS 4B

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OH: W 10AM-12PM (HWP)

Note: we have a EECS16A discord server! Check out piazza post 421

Anonymous feedback form: bit.ly/mosesfb

Vocab needed

- Vector space
- Subspace
- Basis
- Columnspace
- Nullspace
- Dimension

Learning Objectives

1. Finding columnspace and nullspace using Gaussian Elimination on $A\vec{x}=\vec{0}$
   a) Give a basis for it
   b) Express as a span of vectors
2. How to check a set of vectors is a basis for a vectorspace V
   1. Check linear independence
   2. Check that it spans V
3. If time: how to check if a set is a subspace
   1. Closed under scalar multiplication
   2. Closed under vector addition

EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 4B

1. Exploring Column Spaces and Null Spaces

$\underset{m \times n}{A} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

$C(A) = \text{Span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$

$= \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

For the following matrices, answer the following questions:

i. What is the column space of **A**? What is its dimension?

ii. What is the null space of **A**? What is its dimension?

iii. Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same?

sometimes yes

iv. Do the columns of **A** form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

a

$C(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} \stackrel{\checkmark}{=} \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\alpha = 0$

$N(A) = \{\vec{x} \mid \underline{A\vec{x} = \vec{0}}\}$ 1 To find nullspace $N(A)$

Solve $A\vec{x} = \vec{0}$ using G.E.

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$ $x_1 = 0$

$x_2 =$ can be anything

$1 \cdot x_1 + 0 \cdot x_2 = 0$ $x_2 = t$

$\vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} = \underline{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} t \leftarrow$ solution

2 Take the vectors being multiplied by free variables in the solution $\vec{x}$

a $N(A) = \text{Span}\{\text{vectors}\}$

b $N(A)$ has a basis $\{\text{vectors}\}$

$N(A) = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$, $N(A)$ has basis $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$\dim(N(A)) = 1$ why? $\longrightarrow$ basis has only one vector

Revisit: Finding $C(A)$

$A\vec{x} = \vec{0}$

$\hookrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$

end → pivots

Are the columns of A linearly dependent?

$\vec{x} \neq \vec{0}$ then yes

To get a basis for $C(A)$

⇒ choose columns in original matrix that correspond to pivots

$\text{Dim}(C(A)) = 1$

Vectorspace
Subspace
Columnspace
Nullspace

Basis
Dimension

Vectorspace (eg. $\mathbb{R}^n$)

- has a → basis → determines → dimension
- is a → Subspace: subset of a vectorspace that is also itself vectorspace
- is a → columnspace: all vectors $\vec{y} = A\vec{x}$ you can get for any $\vec{x}$ (output of $A\vec{x}$)
- is a → nullspace: all vectors $\vec{x}$ that make $A\vec{x} = \vec{0}$ (Input that makes output $\vec{0}$)

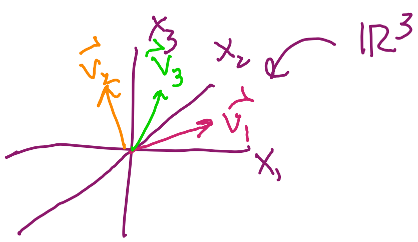
vectorspaces associated a matrix, A

Vectorspace has a "size"

Dimension $\quad \mathbb{R}^n \rightarrow$ n-dimensional space

Basis → set of vectors (minimum set) to generate a vector space

size of the basis set is the dimension of a vector space



$\vec{w} \in \mathbb{R}^3$

$\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3 = \underline{\vec{w}}$

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$

ⓒ $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

→ i. $C(A)$, dim?

→ ii $N(A)$, dim?

→ iii $C(A) \overset{?}{=} C(RREF(A))$

iv Columns a basis for $\mathbb{R}^2$?

$N(A)$: $A\vec{x} = \vec{0}$

$\left[\begin{array}{cccc|c} 1 & -1 & -2 & -4 & 0 \\ 1 & 1 & 3 & -3 & 0 \end{array}\right]$

$\downarrow$ GE

$\left[\begin{array}{cccc|c} 1 & 0 & 1/2 & -7/2 & 0 \\ 0 & 1 & 5/2 & 1/2 & 0 \end{array}\right]$

$RREF(A) = \begin{bmatrix} 1 & 0 & 1/2 & -7/2 \\ 0 & 1 & 5/2 & 1/2 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$x_3 = s$

$x_4 = t$

$x_1 + \frac{1}{2}s - \frac{7}{2}t = 0$

$x_2 + \frac{5}{2}s + \frac{1}{2}t = 0$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s + \frac{7}{2}t \\ -\frac{5}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix}$

$\vec{x} = \begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} t$

$\boxed{\text{Dim}(N(A)) = 2}$

$\boxed{\text{Basis for } N(A) = \left\{ \begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix} \right\}}$

$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \vec{x} = \vec{w} \in \mathbb{R}^3 \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

2. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? If the vectors do not form a basis for $\mathbb{R}^3$, can they be thought of as a basis for some other vector space? If so, write an expression describing this vector space.

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \to \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Defn:
{Basis} of vectorspace V
1. Span{basis} = V
2. {basis} has to be Linearly independent

Does V_1 span $\mathbb{R}^3$? Y / N — can't form vectors like $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{V_1 \text{ is not a basis}}$

Does V_2 span $\mathbb{R}^3$? Y / N

Is V_2 linearly independent? Y / N

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\boxed{V_2 \text{ is a basis}}$ ✓

$\boxed{V_3 \text{ not a basis}}$ ✓ Why?: Not linearly independent $\to \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Q: When can $C(A) \neq C(RREF(A))$?

Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad RREF(A) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad C(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$? $\quad C(RREF(A)) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$C(A)$ $\quad A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$

$C(A)$ has a basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ $\quad \dim(C(A)) = 2 \quad C(A) = \mathbb{R}^2$

$RREF(A) = \begin{bmatrix} 1 & 0 & 1/2 & -7/2 \\ 0 & 1 & 5/2 & 1/2 \end{bmatrix}$

$N(A)$ has a basis $\begin{bmatrix} -1/2 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7/2 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$

$-\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \vec{0}$

3. (Optional Practice) Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = \underline{c} \begin{bmatrix}1\\1\\1\end{bmatrix} + \underline{d} \begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

① Check that V is closed under scalar multiplication

$\vec{v} \in V$ and $\alpha \in \mathbb{R}$ V is closed " " if $\alpha\vec{v} \in V$

$$\vec{v} \in V \rightarrow \vec{v} = c_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + d_1 \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\alpha\vec{v} = \alpha \left(c_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + d_1 \begin{bmatrix}1\\0\\1\end{bmatrix} \right)$$

$$= \underline{\alpha c_1} \begin{bmatrix}1\\1\\1\end{bmatrix} + \underline{\alpha d_1} \begin{bmatrix}1\\0\\1\end{bmatrix}$$

Define $c_2 = \alpha c_1$ $d_2 = \alpha d_1$

$$\alpha\vec{v} = \underline{c_2} \begin{bmatrix}1\\1\\1\end{bmatrix} + \underline{d_2} \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$\alpha\vec{v} \overset{\checkmark}{\in} V$ V is closed under sca. multi.

② Check that V is closed under vector addition

$\vec{v}_1 \in V$ and $\vec{v}_2 \in V$ want to see if $\vec{v}_1 + \vec{v}_2 \in V$

$$\vec{v}_1 = c_1 \underline{\begin{bmatrix}1\\1\\1\end{bmatrix}} + d_1 \begin{bmatrix}1\\0\\1\end{bmatrix} \qquad \vec{v}_2 = c_2 \underline{\begin{bmatrix}1\\1\\1\end{bmatrix}} + d_2 \begin{bmatrix}1\\0\\1\end{bmatrix}$$

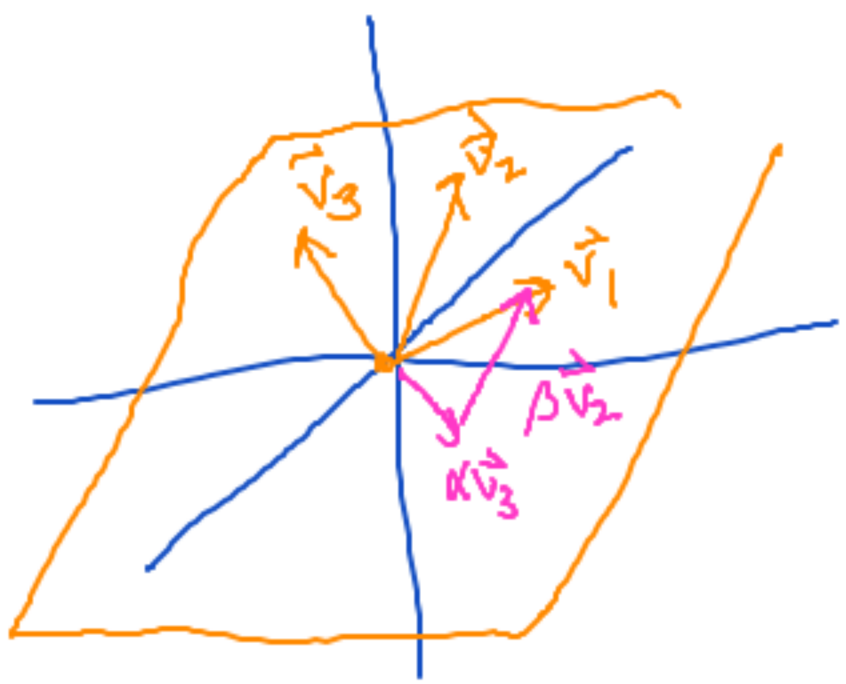
$$\vec{v}_1 + \vec{v}_2 = \underbrace{(c_1 + c_2)}_{c_3} \underline{\begin{bmatrix}1\\1\\1\end{bmatrix}} + \underbrace{(d_1 + d_2)}_{d_3} \underline{\begin{bmatrix}1\\0\\1\end{bmatrix}} \in V \Rightarrow$$ V is closed under vec. add.

V is a subspace of $\mathbb{R}^3$ ✓ vectors in V came from $\mathbb{R}^3$ (vectors have 3 entries)

Q: What is linear dep / indep ?

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \rightarrow$ linearly dependent if I have redundant directions

$\rightarrow$ some vectors $\vec{v}_i$ can be made using the other vectors



Linearly independent set of vectors:

no vector can be formed by a lin. comb. of the other vectors

$\rightarrow$ All directions are "unique" not redundant

Q: Examples of linearly dependent and independent sets of vectors

A: LD $\rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$ LI $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ LI $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

Defn: LD ① $\{\vec{v}_1, \dots, \vec{v}_n\}$ is LD if $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0}$

(at least one $\alpha_i \neq 0$)

② " " " if $\underline{\vec{v}_i} = \beta_1\vec{v}_1 + \cdots + \beta_{i-1}\vec{v}_{i-1} + \beta_{i+1}\vec{v}_{i+1} + \cdots \beta_n\vec{v}_n$

LI ① Not LD

ⓐ We can't get $\vec{0}$ unless we do: $0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_n = \vec{0}$

ⓑ No vector $\vec{v}_i$ can be formed as a linear combination of the other $\vec{v}_j$ $(j \neq i)$