EECS 16A DIS 4~~B~~ A

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OH: W 10AM-12PM (HWP)
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Questions while we wait for Berkeley time

Learning Objectives

① More computing inverses practice
Term: The method of finding inverses using row operations on $[A \mid I] \hookrightarrow [I \mid A^{-1}]$ is known as Gauss-Jordan Elimination

② Another pumps problem in preparation of a concept to be shown in lecture this week.
- transition matrix practice, matrix-matrix multiplication practice

③ If time: practice question: Columnspace + Nullspace preview*

Lecture note references: Lecture 2A pg 4
Lecture 3B pg 7-8

Note references: Note 7
Note 8

* Not fully covered in lecture, more to come tomorrow

EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 4A

1. Mechanical Inverses

In each part, determine whether the inverse of **A** exists. If it exists, find it.

Q: How to do this?

A: Look at a specific quantity ad-cb

A: Just reduce $[A \mid I]$, $AX = I$

A: Check if columns of A are linearly dependent

↳ $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{0}$ if at least one $x_i \neq 0$

$$\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{0}, \quad \vec{x} \neq \vec{0} \Rightarrow \text{linearly dependent}$$

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/9 \end{array}\right]$$

A I I A^{-1}

$$\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Way to check: do the matrix matrix-mult.

(b) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{b}{a}c & -\frac{c}{a} & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a \cdot 1}{a(d-\frac{b}{a}c)} \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a}+\frac{bc}{a(ad-bc)} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\frac{1}{a} + \frac{bc}{a(ad-bc)} = \frac{ad-\cancel{bc}}{a(ad-bc)} + \frac{\cancel{bc}}{a(ad-bc)} = \frac{d}{ad-bc}$$

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

formula for 2x2 inverse

(c) (PRACTICE)

$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$ → Invertible, practice yourself

(d) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

Not invertible because columns are linearly dependent

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/5 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2, R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/5 & 0 & 0 \\ 0 & 0 & -2 & -2/5 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/5 & 0 & 0 \\ 0 & 0 & 1 & 1/5 & -1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \Rightarrow \text{linearly dependent}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/5 & 0 & 0 \\ 0 & 0 & 1 & 1/5 & 1 & 0 \\ 0 & 0 & 0 & -6/5 & 1/2 & 1 \end{array}\right]$$

$AX = I$

$A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ldots$

$A\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\left.\begin{array}{l} 0 = -6/5 \\ 0 = 1/2 \\ 0 = 1 \end{array}\right\} \text{contradictory} \Rightarrow \text{Not invertible}$$

(e) (PRACTICE)

$\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix}$

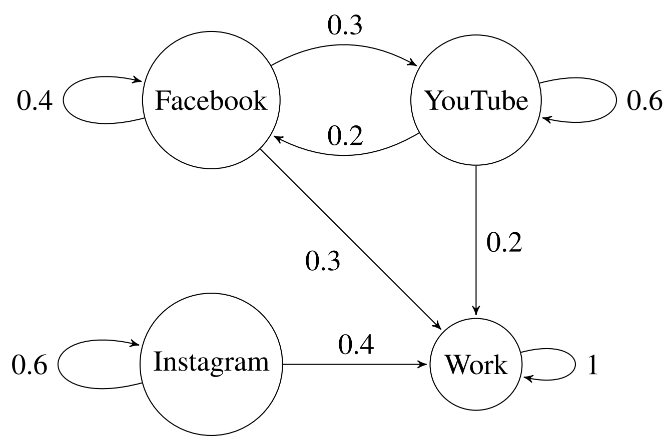
↳ no obviously linear dependent columns good chance it's invertible

2. Social Media

As a tech-savvy Berkeley student, the distractions of social media are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Facebook or YouTube? How do other students manage to get stuff done and balance pursuing Insta-fame? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if 100 students are on Facebook, in the next timestep, 30 of them will click on a link and move to YouTube.

Pumps Word prob.

① Define the state



$$\vec{X}[n] = \begin{bmatrix} X_{FB}[n] \\ X_{YT}[n] \\ X_{I}[n] \\ X_{W}[n] \end{bmatrix}$$

State time n.

② Write equations + write transition matrix

(a) Derive the corresponding transition matrix.

(b) There are 1500 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 700 EECS16A students on Facebook, 450 on YouTube, 200 on Instagram, and 150 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

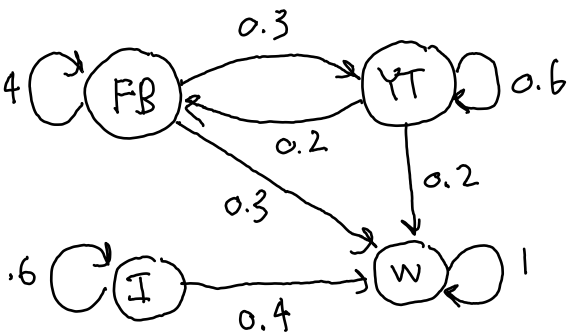
(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

(d) You want to predict how many students will be on each website n timesteps in the future. How would you formulate that mathematically? Without working it out, can you predict roughly how many students will be in each state 100 timesteps in the future?

(e) **Challenging Practice Problem:** Suppose, instead of having 'Work' as an explicit state, we assume that any student not on Facebook/Youtube/Instagram is working. Work is like the "void," and if a student is "leaked" from any of the other states, we assume s/he has gone to work and will never come back. How would you reformulate this problem? Redraw the figure and rewrite the appropriate transition matrix. What are the major differences between this problem and the previous one?

$\boxed{2}$ (a) Derive the transition matrix

$$\begin{bmatrix} x_{FB}[n+1] \\ x_{YT}[n+1] \\ x_{I}[n+1] \\ x_{W}[n+1] \end{bmatrix} = \begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} x_{FB}[n] \\ x_{YT}[n] \\ x_{I}[n] \\ x_{W}[n] \end{bmatrix}$$



$x_{FB}[n+1] = 0.4\, x_{FB}[n] + \cdots +$

sum of things leaving

$1.0\, x_{FB}[n] - (0.3\, x_{FB}[n] + 0.3\, x_{FB}[n])$

(b) How many people doing each activity at next time?

Right now: 700 on FB, 450 on YT, 200 on I, 150 at W

$$\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.3 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0.3 & 0.2 & 0.4 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 700 \\ 450 \\ 200 \\ 150 \end{bmatrix}}_{1500} = \begin{bmatrix} 370 \\ 480 \\ 120 \\ 530 \end{bmatrix} \rightarrow \underline{1500}$$

(c) Compute column sum. Interpretation?

Since all columns sum to 1, the total # of students everywhere should stay the same (conservative system)

(d) How many students in each 100 timesteps into the future?

Can we predict without computing? → Yes, all students should end @ work

$T, \vec{x}[n]$ $\quad\quad$ $\vec{x}[n+100]$?

$\vec{x}[n+1] = T\vec{x}[n]$

$\vec{x}[n+2] = T\vec{x}[n+1] = T(T\vec{x}[n]) = T^2 \vec{x}[n]$

$\Rightarrow \vec{x}[n+100] = T^{100}\vec{x}[n] \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1500 \end{bmatrix}$

$$\lim_{m \to \infty} \vec{x}[n+m] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1500 \end{bmatrix}$$

(e) What if no work?

- Transition matrix?
- Diagram?
- Behavior?

Q: Do we get everyone in work in 100 timesteps?

A: No not exactly. Only in the limit

3. Practice: Column Spaces and Null Spaces Intro

- The **column space** is the possible outputs of a transformation/function/linear operation. It is also the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?

ii. What is the null space of $\mathbf{A}$? What is its dimension?

iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?

iv. Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^2$? Why or why not?

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

Q: How to tell if cols. lin. in/dependent?

$n \times 2$ matrix : $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ $\vec{v}_1 = \alpha \vec{v}_2 \Rightarrow \vec{v}_1 - \alpha \vec{v}_2 = \vec{0}$

A: If 2 columns only, check if scalar mult.

If so $\Rightarrow$ $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent

If not scalar mult. $\Rightarrow$ $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent

Q: How about for more cols?

A: $\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \vec{x} = \vec{0}$ If $\vec{x} \neq \vec{0}$ is a solution, then $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ is LD

G.E. $\left[\begin{array}{cccc|c} | & | & & | & \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{0} \\ | & | & & | & \end{array}\right] \longrightarrow \vec{x} = ?$

$\nearrow$ $\vec{x} = \vec{0}$ LI

$\searrow$ $\vec{x} \neq \vec{0}$ LD