EECS16A DIS 3B

Email: moseswon@berkeley.edu

OH : W 10AM-12PM PST

Anonymous feedback form: bit.ly/mosesfb

Any questions while we wait for Berkeley time?

Learning objectives

1. When is a matrix-matrix product computable? + Other properties
2. Pump systems and how to write their equations/ how to draw them
   a. Drawing → equations → transition matrix
   b. Equations → Drawing
3. Transpose operation on a matrix
4. Practice matrix inverse computation. (example)
5. Slight introduction to notion of conservation (conservative system)

EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 3B

1. Matrix Multiplication

$a_1b_1 + a_2b_2 + \cdots + a_nb_n$

$\sum_{i=1}^{n} a_ib_i = [a_1\ a_2 \cdots a_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ if you can compute

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

$\rightarrow A\vec{b}$ ✓
$\rightarrow AB$ ✓

For each matrix multiplication problem, *if the product exists*, find the product by hand. Otherwise, explain why the product does not exist.

When is matrix-matrix multiplication possible

$\underset{m\times n}{A}\ \underset{p\times q}{B} = \underset{m\times q}{C}$ (rows cols, rows cols)

if $n = p$

$\underset{m\times n\ \ n\times q}{AB} = \underset{m\times q}{C}$

(a) **A B**

$\underset{1\times 2}{[1\ 4]}\underset{2\times 1}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}} = 1\cdot 3 + 4\cdot 2 = \underset{1\times 1}{11}$

(b) **C D**

$\vec{d}_1\ \vec{d}_2$

$\underset{2\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}}\underset{2\times 2}{\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}} = \begin{bmatrix} C\vec{d}_1 & C\vec{d}_2 \end{bmatrix}$

CD is computable

$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 4\cdot 2 & 1\cdot 2 + 4\cdot 1 \\ 2\cdot 3 + 3\cdot 2 & 2\cdot 2 + 3\cdot 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \\ 12 & \cancel{8}\ 7 \end{bmatrix}$

(c) **D C**

$2\times 2\ 2\times 2$ ✓

$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3+4 & 12+6 \\ 2+2 & 8+3 \end{bmatrix} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}$

$\neq$

$\Rightarrow$ Matrix-Matrix mult. is not commutative

$AB \neq BA$

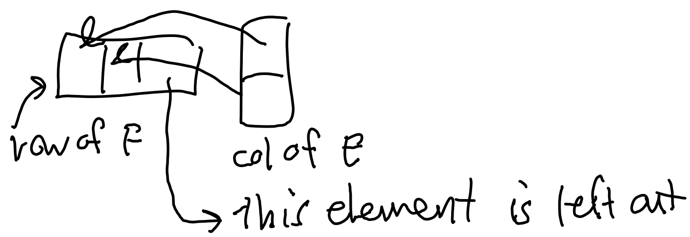
(d) **C E**

2x2 2x4
$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}$$
2x4

→ (e) **F E**

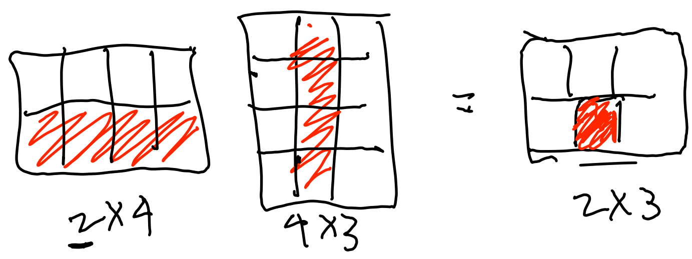
4x3 2x4

No! Why:



→ (f) **E F**

2x4 4x3



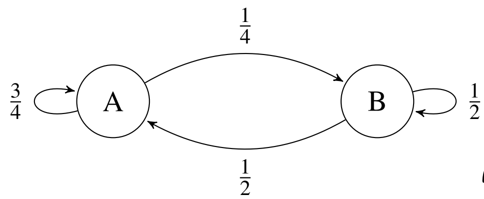
(g) **G H** (Practice on your own)

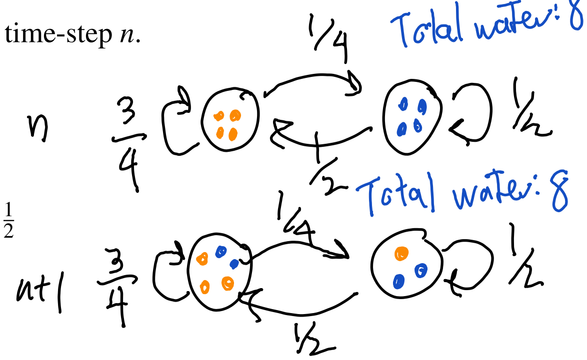
(h) **H G** (Practice on your own)

2. Transition Matrix

Suppose we have a network of pumps as shown in the diagram below. Let us describe the state of A and B using a state vector $\vec{x}[n] = \begin{bmatrix} x_A[n] \\ x_B[n] \end{bmatrix}$ where $x_A[n]$ and $x_B[n]$ are the states at time-step n.

$$S\begin{bmatrix}4\\4\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix}$$





(a) Find the state transition matrix S, such that $\vec{x}[n+1] = S\,\vec{x}[n]$.

$$\to\ x_A[n+1] = \frac{3}{4}x_A[n] + \frac{1}{2}x_B[n]$$

$$\to\ x_B[n+1] = \frac{1}{4}x_A[n] + \frac{1}{2}x_B[n]$$

$$\begin{bmatrix}x_A[n+1]\\x_B[n+1]\end{bmatrix} = \begin{bmatrix}3/4 & 1/2\\1/4 & 1/2\end{bmatrix}\begin{bmatrix}x_A[n]\\x_B[n]\end{bmatrix} = \begin{bmatrix}\square\\\square\end{bmatrix}$$

Transition matrix

$$S = \begin{bmatrix}3/4 & 1/2\\1/4 & 1/2\end{bmatrix}$$

When columns sum to 1 + conservative system (total substance doesn't change)

(b) Let us now find the matrix S^{-1} such that we can recover the previous state $\vec{x}[n-1]$ from $\vec{x}[n]$. Specifically, solve for S^{-1} such that $\vec{x}[n-1] = S^{-1}\,\vec{x}[n]$.

Q: Why do we use I in [S|I] to find S^{-1}

A: $\underset{2\times2}{S}\,\underset{2\times2}{X} = \underset{2\times2}{I}$

$A\vec{x} = \vec{b}$

$[A\,|\,\vec{b}]$

To find inverse: Use GE

$$[\,S \mid \underset{2\times2}{I}\,] \xrightarrow{\frac{R_1}{3/4}\to R_1} \left[\begin{array}{cc|cc}1 & \frac{2}{3} & \frac{4}{3} & 0\\ 1/4 & 1/2 & 0 & 1\end{array}\right]$$

$$\left[\begin{array}{cc|cc}3/4 & 1/2 & 1 & 0\\ 1/4 & 1/2 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_2 - 1/4R_1\to R_2}\left[\begin{array}{cc|cc}1 & 2/3 & 4/3 & 0\\ 0 & \underbrace{\frac{1}{2}-\frac{1}{6}}_{1/3} & -\frac{1}{3} & 1\end{array}\right] \xrightarrow{3R_2\to R_2} \left[\begin{array}{cc|cc}1 & 2/3 & 4/3 & 0\\ 0 & 1 & -1 & \cancel{1}\ 3\end{array}\right] \xrightarrow{R_1-\frac{2}{3}R_2\to R_1} \left[\begin{array}{cc|cc}1 & 0 & 2 & -\cancel{2/3}\ 2\\ 0 & 1 & -1 & \cancel{1}\ 3\end{array}\right]$$

$$\boxed{S^{-1} = \begin{bmatrix}2 & -2\\-1 & 3\end{bmatrix}}$$

(c) Now draw the state transition diagram that corresponds to the S^{-1} that you just found.

Start (Matrix-vector)

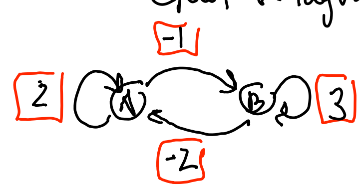
$$\begin{bmatrix}x_A[n-1]\\x_B[n-1]\end{bmatrix} = \begin{bmatrix}2 & -2\\-1 & 3\end{bmatrix}\begin{bmatrix}x_A[n]\\x_B[n]\end{bmatrix}$$

Equations

$$\underbrace{x_A[n-1]}_{} = 2x_A[n] - 2x_B[n]$$

$$\underbrace{x_B[n-1]}_{\text{what we get}} = \underbrace{-x_A[n] + 3x_B[n]}_{\text{what contribute}}$$

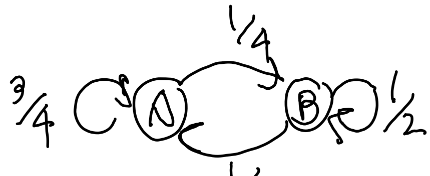
Goal (Diagram)



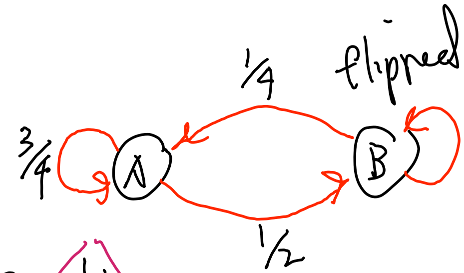
$$\begin{bmatrix}x_A[n]\\x_B[n]\end{bmatrix} \to x_A[n]+x_B[n] = x_A[n-1]+x_B[n-1]$$

Q: Do inverses exist for non square matrices? A: In 16A No

(d) Redraw the diagram from the first part of the problem, but now with the directions of the arrows reversed. Let us call the state transmission matrix of this "reversed" state transition diagram T. Does $T = S^{-1}$?



$S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$ Started



$S^T = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$ (S-transpose)

$\begin{bmatrix} \vec{v} \end{bmatrix}$ $\begin{bmatrix} \vec{v}^T \end{bmatrix}$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\vec{v}^T = [v_1\ v_2\ v_3]$

(e) Suppose we start in the state $\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$. Compute the state vector after 2 time-steps $\vec{x}[3]$.

Everything here and below

Not shown during recording

$\vec{x}[3] = S\vec{x}[2]$

$\vec{x}[2] = S\vec{x}[1]$

$\Rightarrow \vec{x}[3] = S(S\vec{x}[1])$

Matrix-Matrix mult. is associative:

$(SS)\vec{x}[1] = S(S\vec{x}[1])$

compute $S \cdot S$ first

compute $S\vec{x}[1]$ first

Another example of transpose

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

flip along diagonal, rows become columns, columns become rows

(f) **(Challenge practice problem)** Given our starting state from the previous problem, what happens if we look at the state of the network after a lot of time steps? Specifically which state are we approaching, as defined below?

$$\vec{x}_{final} = \lim_{n\to\infty} \vec{x}[n]$$

Note that the final state needs to be what we call a *steady state*, meaning $S\,\vec{x}_{final} = \vec{x}_{final}$.

Also what can you say about $x_A[n] + x_B[n]$?

Use information from both of these properties to write out a new system of equations and solve for $\vec{x}_{final}$.

$S\vec{x}_{final} = \vec{x}_{final}$ $\vec{x}_{start} = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$

S is conservative i.e. $\boxed{x_{A final} + x_{B final}} = x_{A start} + x_{B start} = 24$

So: $S\vec{x}_{final} - I\vec{x}_{final} = \vec{0}$

matrix matrix mult is distributive (but shown in reverse can factor!)

$(S - I)\vec{x}_{final} = \vec{0}$

$$\boxed{\left[\begin{array}{cc|c} \frac{3}{4}-1 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2}-1 & 0 \end{array}\right]}$$

$$\left[\begin{array}{cc|c} -1/4 & 1/2 & 0 \\ 1/4 & -1/2 & 0 \\ 1 & 1 & 24 \end{array}\right]$$

$\boxed{\begin{aligned} x_{A final} &= 16 \\ x_{B final} &= 8 \end{aligned}}$

Q: What is the point of transpose?

A: Notationally helpful. More ideas in 16B, later in 16A

Q: Can we have arrows coming out from some place not add up to 1?

A: Yeah! Just not conservative

Example

$\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} \vec{a}_1^T\vec{b}_1 & \cdots \\ \vdots & \ddots \\ \vec{a}_n^T\vec{b}_1 & \cdots\ \vec{a}_n^T\vec{b}_n \end{bmatrix}$

$\vec{a}_i^T\vec{b}_j = \underbrace{[\ \ \cdots\ \ ]}_{n} \begin{bmatrix} \vdots \end{bmatrix} \}n$