EECS16A Dis 3A

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New OH time (mine + other TAs/ASEs) : W 10AM-12PM (HW party but conceptual Q's welcome)

OH / HWP: oh.eecs16a.org

Today's topics / takeaways

1. Proof example
2. Matrices as geometric operations : Rotation matrix (2x2)
3. Whether Operations are commutative

EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 3A

Suggestions on problem prioritization for when doing individual work or collaborating with groupmates:

(a) Ask the TA to walk through 1(a) or limit time on 1(a).

(b) Work through 2 parts 1 and 2 for the rest of the time remaining.

(c) Try 1(b) as practice after having seen 1(a).

1. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, show the following:

(a)

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

(b) **(Practice)**

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!

Part 1: Rotation Matrices as Rotations

(a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by 15° and 30°, respectively. Design a procedure to rotate the unit square by 45° using only $\mathbf{T}_1$ and $\mathbf{T}_2$, and plot the result in the IPython notebook. How would you rotate the square by 60°?

(b) Try to rotate the unit square by 60° using only one matrix. What does this matrix look like?

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called "rotation matrices." They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. To do this consider rotating the unit vector $\begin{bmatrix} cos(\alpha) \\ sin(\alpha) \end{bmatrix}$ by θ degrees using the matrix $\mathbf{R}$.

1 ⓐ Given a set of n vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

Show that $\underline{\text{span}} \underbrace{\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}}_{②} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}, \alpha \neq 0$

set of all
Span ~ solutions to $\underline{A\vec{x} = \vec{b}}$

→ set of all linear combinations of a set of vectors ②

$\vec{p} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$

$\vec{p} = \beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \dots + \beta_n\vec{v}_n$

$\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = \{1\cdot\vec{v}_1 + \dots + 0\vec{v}_n,\ 1.5\vec{v}_1 + \dots + 3\vec{v}_n,\ \vdots\ \}$

We want to show that two sets are equal

$\{a, b, c\} \overset{?}{\neq} \{a, b, c, d\}$

Not equal because an object in second set does not appear in first

Insight: for sets to be equal every thing in first set must be in second set and vice versa

Proof steps *

① Read the statement
↳ What are we being asked to show?

② Write what you know (in notation) (starting point)

③ Write what you want to prove (also in notation) (end point/goal)

④ Similarities in ② + ③

⑤ Simple example

⑥ Manipulate both start + end. Justify each step.

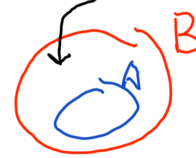
⑦ Try other proof approaches

* Findable in note 4, reduced steps in "Proof notes" on 9/10 block in schedule on website

$(\square)\ \underline{\vec{p} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}} \Rightarrow \underline{\vec{p} \in \text{Span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}}$

$(\Delta)\ \underline{\vec{q} \in \text{Span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}} \Rightarrow \underline{\vec{q} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}}$

subset
$(\square)\ A \subseteq B$



$(\Delta)\ B \subseteq A$

$A = \text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$ set

$B = \text{Span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ set

Prove $(\square)$

$\vec{p} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$

(definition)

☆ $\vec{p} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \cdots + \beta_n \vec{v}_n$ by defn. of span (justification)

Want that $\vec{p} \in \text{Span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

☆ $\vec{p} = \gamma_1(\alpha\vec{v}_1) + \gamma_2 \vec{v}_2 + \cdots + \gamma_n \vec{v}_n$

Choose γ_i. $\gamma_2 = \beta_2$ $\qquad \gamma_1 \alpha = \beta_1$

$\gamma_3 = \beta_3$ $\qquad \gamma_1 = \frac{\beta_1}{\alpha}$

$\vdots$

$\gamma_n = \beta_n$

(justification)

$\vec{p} = \frac{\beta_1}{\alpha}(\alpha\vec{v}_1) + \beta_2 \vec{v}_2 + \cdots + \beta_n \vec{v}_n$ because $\left\{ \begin{array}{l} \text{of rules of mult./div.} \\ \frac{\alpha}{\alpha} = 1 \end{array} \right.$

$\vec{p} \in \text{Span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ by defn.

Prove $(\triangle)$

$\vec{q} \in \text{Span}\{\alpha\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \Rightarrow \vec{q} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$

$\vec{q} = x_1(\alpha\vec{v}_1) + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n$ by definition

$\vec{q} = \cancel{(\alpha x_1)} \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n$ (because multiplication is associative)

$(x_1 \alpha)$

$\vec{q}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_n$

$\vec{q} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ by defn.

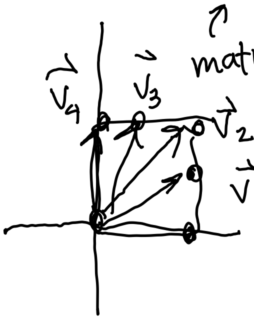
Since $(\square), (\triangle)$: The sets must be equal $\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\} = \text{Span}\{\alpha\vec{v}_1, \dots, \vec{v}_n\}$

$\boxed{2}$ (a) If T_1 rotates by 15° (CCW) and T_2 rotates by 30° (CCW) how to rotate by 45° (CCW) or 60° (CCW)?

CCW ↳ counter clockwise

Q: What does it mean to rotate a square / how are we using matrices T_1, T_2?

A: $\underset{\text{matrix}}{T_1}\,\underset{\text{vector}}{\vec{v}} = \vec{w} \leftarrow \vec{v}$ after it's been rotated by 15°



$T_1\vec{v}_i = \vec{w}_i \leftarrow$ point of a square that has been rotated

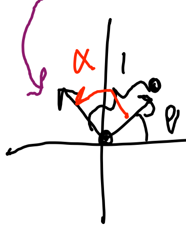
$T_1\vec{v} \to 15^\circ$ rotated

$T_2(T_1\vec{v}) \to 15^\circ$ rotated then 30° rotated $\to 45^\circ$ rotated

$$[\quad]_{2\times 2}[\quad]_{2\times 1}$$

$\underbrace{T_1\ T_2}\vec{v} \to$ another way to get 45°

2 ⓒ Show that a rotation matrix (2x2) has form $\boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$



$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$$\begin{bmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{bmatrix} \overset{?}{=} \underset{2\times2}{\begin{bmatrix} \cos\cancel{\theta}\,\alpha & -\sin\cancel{\theta}\,\alpha \\ \sin\cancel{\theta}\,\alpha & \cos\cancel{\theta}\,\alpha \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}$$

$$\overset{?}{=} \begin{bmatrix} \cos\theta\cos\alpha - \sin\theta\sin\alpha \\ \cos\theta\sin\alpha + \sin\theta\cos\alpha \end{bmatrix}$$

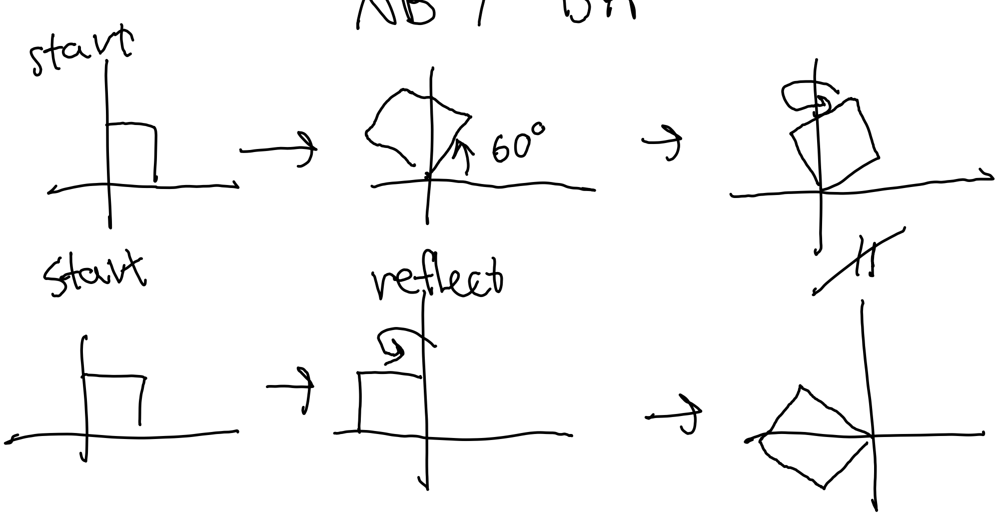
$$\overset{\checkmark}{=} \begin{bmatrix} \cos(\theta+\alpha) \\ \sin(\theta+\alpha) \end{bmatrix}$$ (from trig identities)

2 Part 2

Operations from matrices are not commutative

$$\underset{\text{matrix}}{A}\ \underset{\text{matrix}}{B}\ \vec{v} \neq BA\vec{v}$$

$$AB \neq BA$$



(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \ldots} = 1$.)

(*Hint: Use your trigonometric angle sum identities:* $\cos(\theta_1 + \theta_2) = cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$ *and* $\sin(\theta_1 + \theta_2) = sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)$)

(d) **(Practice)** Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? (**Note:** Don't use inverses! Answer this question using your intuition, we will visit inverses very soon in lecture!)

(e) **(Practice)** Use part (d) to obtain the "inverse" rotation matrix for a matrix that rotates a vector by θ. Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

(f) **(Practice)** What are the matrices that reflect a vector about the (i) x-axis, (ii) y-axis, and (iii) $x = y$

Part 2: Commutativity of Operations

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Follow your TA to obtain the answers to the following questions!

(a) Let's see what happens to the unit square when we rotate the square by 60° and then reflect it along the y-axis.

(b) Now, let's see what happens to the unit square when we first reflect the square along the y-axis and then rotate it by 60°.

(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

(Practice) Part 3: Distributivity of Operations

(a) The distributivity property of matrix-vector multiplication holds for any vectors and matrices. Show for general $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ that $\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2$.