EECS16A DIS 2B

Todays' topics

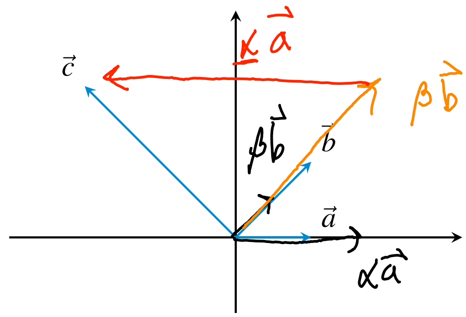
1. Span (of a set of vectors as a geometric concept)
   - a. How to check if a vector is in the span of other vectors
2. Translating a problem into a system of equations
3. If time: More Gaussian elimination examples

If you're present here on zoom: attendance already taken
If you're watching the video recording: fill out checkoff form
(only need to do one or the other)

EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 2B

1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



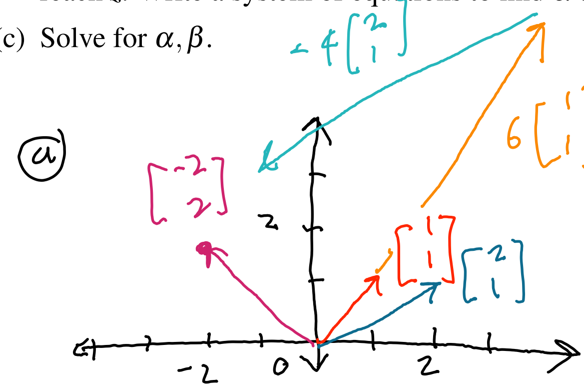
(b) Check if $\vec{z} \in \text{span}\{\vec{x}, \vec{y}\}$

(a) First, consider the case where $\vec{x}=\begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y}=\begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z}=\begin{bmatrix}-2\\2\end{bmatrix}$. Draw these vectors on a sheet of paper.

(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$ so that we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

(c) Solve for α, β.

(a)



How to check a vector is in a span

(b) $\vec{z} = \alpha\vec{x} + \beta\vec{y}$ (not in matrix form)

$$\begin{bmatrix}-2\\2\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$$ (Matrix form)

(c) $$\left[\begin{array}{cc|c}1 & 2 & -2\\\boxed{1} & 1 & 2\end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c}1 & 2 & -2\\0 & \boxed{-1} & 4\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1 & \boxed{2} & -2\\0 & 1 & -4\end{array}\right]$$

REF

$$\rightarrow \boxed{\left[\begin{array}{cc|c}1 & 0 & 6\\0 & 1 & -4\end{array}\right]}$$

RREF

← helps you write solutions

$\alpha = 6$
$\beta = -4$

RREF doesn't always give a square identity matrix on the left

2. Finding The Bright Cave

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoul fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers, but they don't know any linear algebra – and they need your help.

Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

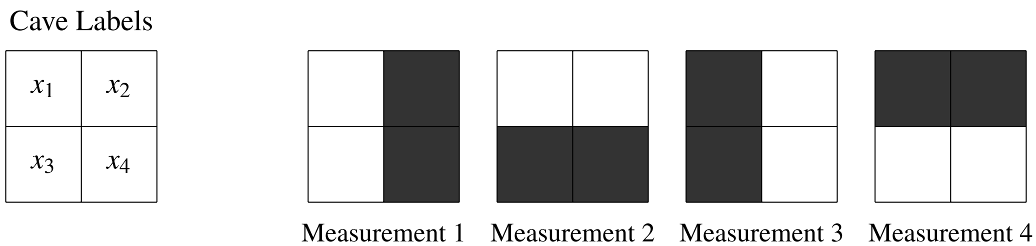


Figure 1: Four image masks.

(a) Let $\vec{x}$ be the four-element vector that represents the magnitude of light emanating from the four cave entrances. Write a matrix $\mathbf{K}$ that performs the masking process in Figure 1 on the vector $\vec{x}$, such that $\mathbf{K}\vec{x}$ is the result of the four measurements.

(b) Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not?

(c) Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However, her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

50%	100%
100%	50%

Does this additional measurement give them enough information to solve the problem? Why or why not?

ⓐ

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leftarrow \text{cave light for cave } i$$

① Defined my unknown

$m_1 = x_1 + 0 + x_3 + 0$

$m_2 = x_1 + x_2 + 0 + 0$

$m_3 = 0 + x_2 + 0 + x_4$

$m_4 = 0 + 0 + x_3 + x_4$

② Write eqns

③ Write in matrix form

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \begin{bmatrix} x_1 + 0 + x_3 + 0 \\ x_1 + x_2 + 0 + 0 \\ 0 + x_2 + 0 + x_4 \\ 0 + 0 + x_3 + x_4 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{\vec{x}} \leftarrow$$

How to find $\vec{x}$? GE

(b)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ \boxed{1} & 1 & 0 & 0 & m_2 \\ 0 & 1 & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2-m_1 \\ 0 & \boxed{1} & 0 & 1 & m_3 \\ 0 & 0 & 1 & 1 & m_4 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & m_1 \\ 0 & 1 & -1 & 0 & m_2-m_1 \\ 0 & 0 & 1 & 1 & m_3-m_2+m_1 \\ 0 & 0 & \boxed{1} & 1 & m_4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \boxed{1} & 0 & m_1 \\ 0 & 1 & \boxed{-1} & 0 & m_2-m_1 \\ 0 & 0 & 1 & 1 & m_3-m_2+m_1 \\ 0 & 0 & 0 & \boxed{0} & m_4-m_3+m_2-m_1 \end{array}\right]$$

last eqn : $\underline{0} = \underbrace{m_4 - m_3 + m_2 - m_1}$

Can't write all x_1 to x_4 (no pivot in last column)

$$\left[\begin{array}{cccc|c} 1 & & & & \vdots \\ & 1 & & & \vdots \\ & & 1 & & \vdots \\ & & & 1 & \vdots \end{array}\right]$$

(c) $m_5 = \frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4$

pivots

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & \vdots \\ 0 & \boxed{1} & -1 & 0 & \vdots \\ 0 & 0 & \boxed{1} & 1 & \vdots \\ 0 & 0 & 0 & 0 & \vdots \\ \frac{1}{2} & 1 & 1 & \frac{1}{2} & \vdots \end{array}\right] \xrightarrow{R_5 \leftarrow R_5 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & \vdots \\ 0 & 1 & -1 & 0 & \vdots \\ 0 & 0 & 1 & 1 & \vdots \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & \vdots \end{array}\right]$$

$R_5 \leftarrow R_5 - R_2$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & \vdots \\ 0 & 1 & -1 & 0 & \vdots \\ 0 & 0 & 1 & 1 & \vdots \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & \frac{3}{2} & \frac{1}{2} & \vdots \end{array}\right] \xrightarrow{R_5 \leftarrow R_5 - \frac{3}{2}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & \vdots \\ 0 & 1 & -1 & 0 & \vdots \\ 0 & 0 & 1 & 1 & \vdots \\ 0 & 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & -1 & \vdots \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & \vdots \\ 0 & 1 & 1 & 0 & \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & \end{array}\right] \xrightarrow{\text{Back substitution}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & \boxed{} \end{array}\right]$$

sum of $m_1, \dots, m_5$ $= 0$

$R_5 \leftarrow -R_5$

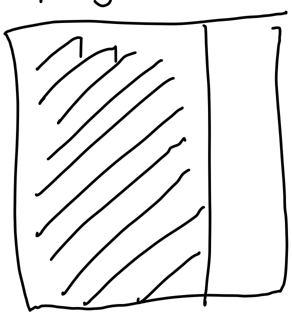
$x_1 =$ something

$x_2 = \dots$

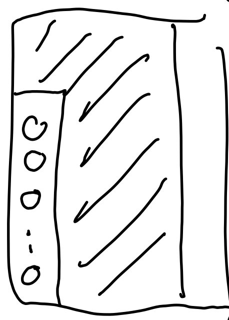
$x_3 = \dots$

$x_4 = \dots$

Augmented



normalize
eliminate
$\rightarrow$



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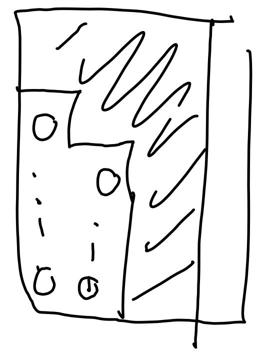

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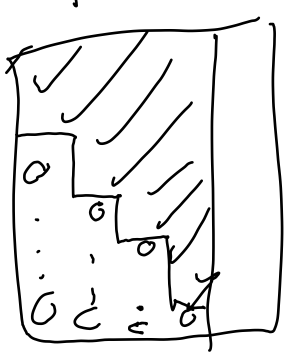
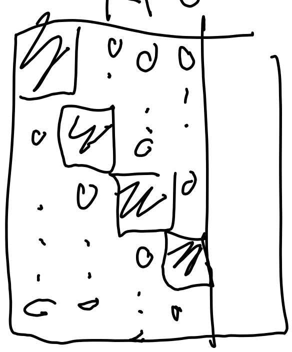


Back
$\rightarrow$

RREF



Question: RREF for non square matrices?

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & \\ 0 & \boxed{1} & 0 & \end{array}\right] \qquad \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & \\ 0 & 0 & \boxed{1} & \end{array}\right]$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & -1 & 0 & \\ 0 & \boxed{1} & 0 & & \end{array}\right]$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & \\ 0 & \boxed{1} & 1 & \end{array}\right]$$

3. (Optional Practice) Gaussian Elimination

Gaussian Elimination is a systematic procedure that a computer could follow for solving a large system of linear equations simultaneously. The augmented matrix is in the form:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \tag{1}$$

Gaussian Elimination Algorithm for augmented matrix (1):

Step 1: For $i = 1, 2, \cdots, m$

(i) If necessary, swap row i with the row below it, so that the leading entry in row i is as far left as possible.

(ii) Rescale row i so that its leading entry is equal to 1.

(iii) For rows $j = i + 1, \cdot, m$, add to row j a scalar multiple of row i, so that the leading entry of row i has all zeros below it.

Step 2: For $i = m, m - 1, \cdots, 1$

(i) Add to each row $j = 1, 2, \cdots, i - 1$ a scalar multiple of row i so that the leading entry of row i has all zeros above it.

For the following systems, use Gaussian Elimination to solve the problem. Does a solution exist? Is it unique? If there are an infinite number of solutions, give the solution in the parametric form.

(a) Let the three variables be x_1, x_2, x_3. Solve the following augmented matrix form using Gaussian Elimination.

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

(b) True or False: A system of equations with more equations than unknowns will always have either infinite solutions or no solutions.

(c)

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 8 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

