EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 5B

1. Steady and Unsteady States

(a) You're given the matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$. ($\vec{x}$ could describe either people or water.) Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$

ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$

iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$$

converges. If it does, what does it converge to?

α	β	γ	Converges?	$\lim_{n\to\infty} \mathbf{M}^n\vec{x}$
0	0	$\neq 0$		
0	$\neq 0$	0		
0	$\neq 0$	$\neq 0$		
$\neq 0$	0	0		
$\neq 0$	0	$\neq 0$		
$\neq 0$	$\neq 0$	0		
$\neq 0$	$\neq 0$	$\neq 0$		

2. Eigenvalues and Special Matrices – Visualization

As seen earlier, an eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

(c) Conceptually, does a rotation matrix in $\mathbb{R}^2$ by angle θ have any eigenvalues $\lambda \in \mathbb{R}$? For which angles is this the case?

(d) Now let us mechanically compute the eigenvalues of the rotation matrix in $\mathbb{R}^2$. Does it agree with our findings above? As a refresher, the rotation matrix $\mathbf{R}$ has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

(e) Does the reflection matrix $\mathbf{T}$ across the x-axis in $\mathbb{R}^{2\times 2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(f) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

(g) **(Practice)** Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

