EECS 16A Designing Information Devices and Systems I
Fall 2020 Discussion 5A

Recall from lecture the way to compute a determinant of any 2×2 matrix is by using the following formula:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(\mathbf{A}) = ad - bc$$

1. Mechanical Eigenvalues and Eigenvectors

Definition: For some matrix $\mathbf{A}$, the polynomial function of λ, $f(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$, is known as the *characteristic polynomial* of $\mathbf{A}$.

Find the eigenvalues (which are the roots of the characteristic polynomial) of each matrix $\mathbf{M}$ and their associated eigenvectors. State if the inverse of $\mathbf{M}$ exists.

(a) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Answer: Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}\right) = 0$$

$$\begin{aligned} -\lambda(-3-\lambda) + 2 &= 0 \\ \lambda^2 + 3\lambda + 2 &= 0 \\ (\lambda+2)(\lambda+1) &= 0 \\ \lambda &= -1, -2 \end{aligned}$$

$\lambda = -1$:

$$\left[\begin{array}{cc|c} 0-(-1) & 1 & 0 \\ -2 & -3-(-1) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = -1$ is $\operatorname{span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

$\lambda = -2$:

$$\left[\begin{array}{cc|c} 0-(-2) & 1 & 0 \\ -2 & -3-(-2) & 0 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ -2 & -1 & 0 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 + x_2/2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t/2 \\ t \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = -2$ is span$\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}$.

Note that we have no zero eigenvalues, the columns of A are linearly independent, and the determinant of A is non-zero (evaluate our polynomial in λ at $\lambda = 0$). Any of these are equivalent conditions for saying that a square matrix is invertible.

(b) $\mathbf{M} = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$

Answer: Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -2-\lambda & 4 \\ -4 & 8-\lambda \end{bmatrix} \right) = 0$$

$$\begin{aligned} (-2-\lambda)(8-\lambda) + 16 &= 0 \\ \lambda^2 - 6\lambda &= 0 \\ \lambda(\lambda - 6) &= 0 \\ \lambda &= 0, 6 \end{aligned}$$

$\lambda = 0$:

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -4 & 8 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = 0$ is span$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

$\lambda = 6$:

$$\left[\begin{array}{cc|c} -2-6 & 4 & 0 \\ 4 & 8-6 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -8 & 4 & 0 \\ -4 & 2 & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - x_2/2 &= 0 \\ x_2 &= t \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/2 \\ t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} t$$

The eigenspace for $\lambda = 6$ is span$\left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}$.

Matrix $\mathbf{M}$ has linearly dependent columns, therefore the inverse $\mathbf{M}^{-1}$ does not exist. Note also that $\mathbf{M}$ has an eigenvalue of 0 so that $N(\mathbf{M})$ contains more than just $\vec{0}$. For this reason also $\mathbf{M}$ is not invertible.

(c) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Answer: Let's begin by finding the eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 = 0$$
$$\lambda = 0(\times 2)$$

$\lambda = 0$:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ cannot be further reduced by G.E.}$$

$$x_2 = 0, x_1 = t \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$$

The eigenspace for $\lambda = 0$ is $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. Note even though $\lambda = 0$ is a eigenvalue with multiplicity 2 (occurs as a root twice for the characteristic polynomial), the dimension of its eigenspace is only 1. In general, it is not possible to find as many linearly independent vectors for as many times a specific eigenvalue occurs for a matrix.
Matrix $\mathbf{M}$ has a zero column (linearly dependent columns), therefore the inverse $\mathbf{M}^{-1}$ does not exist.

(d) $\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Answer: Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$
$$\lambda^2 + 1 = 0$$

From the above equation, we know that the eigenvalues are $\lambda = i$ and $\lambda = -i$.
For the eigenvalue $\lambda = i$:

$$(\mathbf{M} - i\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\vec{x} = \vec{0}$$

We can also perform Gaussian elimination on matrices with imaginary or complex numbers:

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{G.E.} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \begin{array}{c} x_1 - ix_2 = 0 \\ x_2 = t \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} t$$

So the eigenspace is span $\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = -i$:

$$\begin{aligned}
(\mathbf{M} + i\mathbf{I})\vec{x} &= \vec{0} \\
\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} &= \vec{0} \\
\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\right)\vec{x} &= \vec{0} \\
\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\vec{x} &= \vec{0}
\end{aligned}$$

$$\left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \overset{G.E.}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{array}{c} x_1 + ix_2 = 0 \\ x_2 = t \end{array} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} t$$

The second eigenspace is span $\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$.

2. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

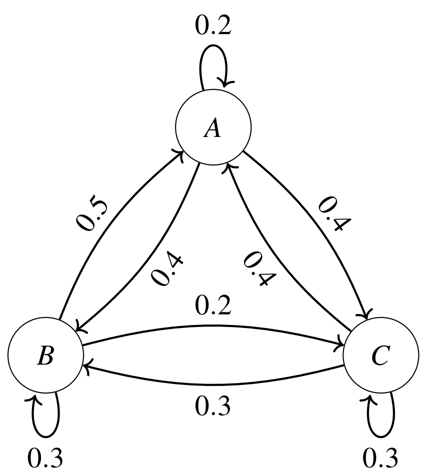

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix $\mathbf{T}$ representing the pumps system.

Answer:

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

(b) You are told that $\lambda_1 = 1$, $\lambda_2 = \frac{-\sqrt{2}-1}{10}$, $\lambda_3 = \frac{\sqrt{2}-1}{10}$ are the eigenvalues of $\mathbf{T}$. Find a steady state vector $\vec{x}$, i.e. a vector such that $T\vec{x} = \vec{x}$.

Answer:

We know $\lambda_1 = 1$ is the eigenvalue corresponding to the steady state eigenvector. Therefore,

$$\begin{aligned}
T\vec{x} &= 1\vec{x} \\
&= \lambda_1 \vec{x} \\
\Rightarrow \vec{x} &\in N(\mathbf{T} - 1 \cdot \mathbf{I}) \\
\vec{x} &\in N\left(\begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\
\vec{x} &\in N\left(\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix}\right).
\end{aligned}$$

In order to row reduce $\mathbf{T} - 1 \cdot \mathbf{I}$ we use Gaussian elimination. We also convert to fractions:

$$\begin{bmatrix} -\frac{4}{5} & \frac{1}{2} & \frac{2}{5} \\ \frac{2}{5} & -\frac{7}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{1}{5} & -\frac{7}{10} \end{bmatrix} \xRightarrow{R_1 \leftarrow -5/4 R_1} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ \frac{2}{5} & -\frac{7}{10} & \frac{3}{10} \\ \frac{2}{5} & \frac{1}{5} & -\frac{7}{10} \end{bmatrix} \xRightarrow[R_3 \leftarrow R_3 - 2/5 R_1]{R_2 \leftarrow R_2 - 2/5 R_1} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ 0 & -\frac{9}{20} & \frac{1}{2} \\ 0 & \frac{9}{20} & -\frac{1}{2} \end{bmatrix}$$

$$\xRightarrow{R_2 \leftarrow -20/9 R_2} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ 0 & 1 & -\frac{10}{9} \\ 0 & \frac{9}{20} & -\frac{1}{2} \end{bmatrix} \xRightarrow{R_3 \leftarrow R_3 - 9/20 R_2} \begin{bmatrix} 1 & -\frac{5}{8} & -\frac{1}{2} \\ 0 & 1 & -\frac{10}{9} \\ 0 & 0 & 0 \end{bmatrix} \xRightarrow{R_1 \leftarrow R_1 + 5/8 R_2} \begin{bmatrix} 1 & 0 & -\frac{43}{36} \\ 0 & 1 & -\frac{10}{9} \\ 0 & 0 & 0 \end{bmatrix}$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a vector describing the steady state, then we can set x_3 to be the free variable. Thus we can write the form any steady state vector should take using the first two equations represented by the row reduced matrix:

$$\begin{aligned} x_1 - \tfrac{43}{36} x_3 &= 0 \\ x_2 - \tfrac{10}{9} x_3 &= 0 \\ x_3 = \alpha &\in \mathbb{R} \end{aligned} \implies \vec{x} = \begin{bmatrix} \frac{43}{36} \\ \frac{10}{9} \\ 1 \end{bmatrix} \alpha$$