CSM 16A Designing Information Devices and Systems I
Fall 2020 Week 11

1. **Properties of Norms**

   Learning Goal: This problem explores various algebraic properties of norms while reinforcing proof skills.

   Relevant Notes: Note 21 goes over inner products, norms, and the Cauchy-Schwarz Inequality (part (b)).

   Prove each of the following theorems using definitions and properties of the inner product and orthogonality.

   (a) **Pythagoras Theorem**
   Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal set of vectors (meaning $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for all $i \neq j$),

   $$\|\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_n\|^2 = \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 + \cdots + \|\vec{v}_n\|^2$$

   (b) **The Cauchy-Schwarz Inequality** [PRACTICE]
   For any vectors $\vec{u}$ and $\vec{v}$,

   $$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$$

   (c) **Triangle Inequality**
   For any vectors $\vec{u}$ and $\vec{v}$,

   $$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

(d) **Mechanical Problems**

(i) Compute the inner product of the two vectors $\begin{bmatrix}5,3,-1\end{bmatrix}^T$ and $\begin{bmatrix}-1,4,-2\end{bmatrix}^T$.

(ii) Compute the norm for each of the vectors given in subpart (i).

(iii) Compute the angle between the vectors.

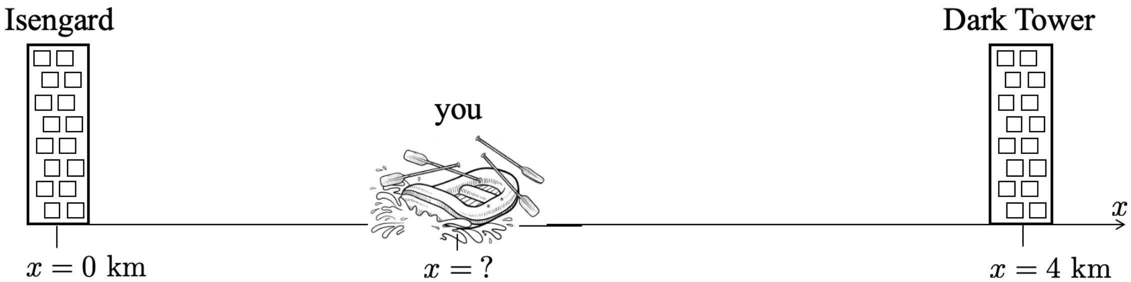
$$\vec{u} = \begin{bmatrix}-1\\4\\0\end{bmatrix}$$
$$\vec{v} = \begin{bmatrix}4\\1\\0\end{bmatrix}$$

2. One Does Not Simply Raft into Mordor

Learning Goal: The goal of this problem is to practice applying cross correlation to word problem scenarios.

Relevant Notes: **Note 22 Section 22.2** covers trilateration, and **Note 22 Section 22.3** covers finding distances with correlation.

You've decided to go rafting to celebrate taking your second midterm! Unfortunately, an hour into your trip, you realize that there are no familiar landmarks nearby, so you're not sure how far you are from your starting point. However, you do remember from your studies of the area that there are two towers: Isengard, at the position $x = 0$ km, and the Dark Tower, at $x = 4$ km. You know you are between the two towers, as shown below:



You recall that each tower emits a sound signal once a day at midday. Specifically, Isengard will emit $\vec{b}_1$ and the Dark Tower will emit $\vec{b}_2$:

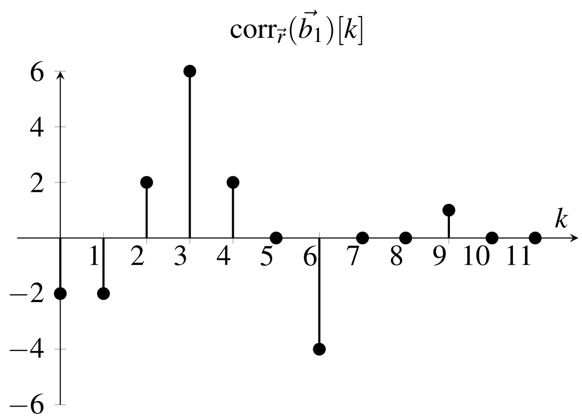
$$\vec{b}_1 = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 \end{bmatrix}^T \qquad \vec{b}_2 = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 \end{bmatrix}^T$$

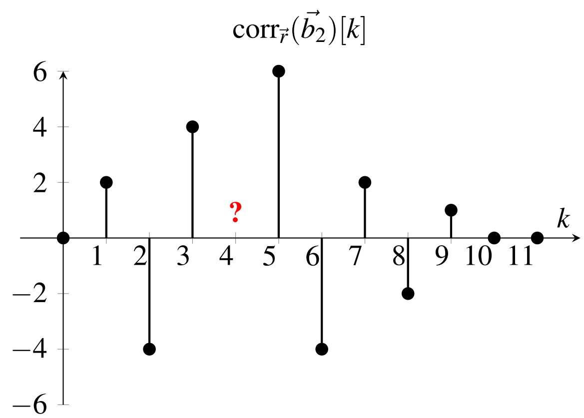
Both signals are emitted at a rate of 2 samples per second (i.e. the sample interval is 0.5 sec), and the signals are emitted only once for each period.

It's only a few minutes from midday so you decide to wait. You use an app on your phone to record the incoming signal (the app also records at 2 samples per second). You start recording at exactly 12:00 PM and receive the following:

$$\vec{r} = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & -2 & 2 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}^T$$

(a) Your first step is to calculate a linear cross-correlation between $\vec{r}$ and each known tower signature. The cross-correlations are plotted below. Calculate the missing value, which is denoted with a question mark.





(b) Recall that the signals were emitted from each building at the same time (12:00 PM). How many seconds after 12:00 PM did it take for the signal from Isengard to reach you? What about the signal from the Dark Tower? Assume that environmental noise (besides the tower-emitted signals) is minimal.

(c) Now, assume that you received Isengard's signal 8 seconds after it was sent and you received the Dark Tower's signal 2 seconds after it was sent. Can you determine your exact position x? If yes, calculate your position. If not, explain why not. Assume sound travels at 340 m/s.

(d) You see a giant eagle, so you get out of your raft to follow it. But you soon realize that you don't know your x position *or* your y position! Luckily, you have a phone app which tells you that you are:

- d_1 km away from Isengard which is located at $x = 0$ km, $y = 0$ km
- d_2 km away from the Dark Tower which is located at $x = 4$ km, $y = 0$ km
- d_3 km away from Minas Tirith which is located at $x = 1$ km, $y = 3$ km

Write a system of linear equations of the form $\mathbf{A}\vec{x} = \vec{b}$ that you can solve to find your position. Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ where x, y have units of kilometers (km).

3. Positioning with Gold Code

Learning Goal: The goal of this problem is to understand how GPS signals are encoded and decoded.

Relevant Notes: **Note 22: Sections 22.3-22.5** walk through the math behind correlations, leading to an overview of the positioning problem.

For this problem, we assume there are 4 GPS satellites in total (In reality, the actual GPS system uses at least 24). Each satellite uses a unique 1024 element long sequence called a Gold code as its "signature." Assume the four satellites use the signatures: $\vec{s_1}$, $\vec{s_2}$, $\vec{s_3}$ and $\vec{s_4}$.

(a) The GPS device receives a 3575 element long signal $\vec{r}$ spanning $n = 0 - 3574$. If we wanted to identify which satellites' signatures are present in $\vec{r}$, what specific correlations should we calculate?

Hint: Recall that the linear correlation of signal $\vec{x}$ with signal $\vec{y}$ is given as:

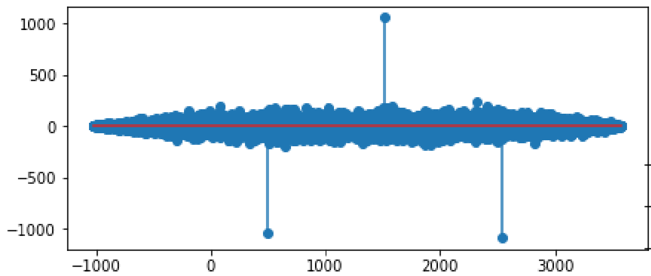
$$corr_{\vec{x}}(\vec{y})[k] = \sum_{n=-\infty}^{\infty} \vec{x}[n]\vec{y}[n-k]$$

.

(b) For each of the correlation operations from part (a), what should be the finite range of time shift k? Remember that $\vec{r}$ has 3575 elements.

(c) Each satellite signature is a sequence of 1024 elements, where each element is either $+1$ or -1. Assuming there is no noise, what would be approximate peak value of $\text{corr}_{\vec{r}}(\vec{s_1})$, if $\vec{s_1}$ and $\vec{s_2}$ both are present in $\vec{r}$?

(d) The following figure shows the results of cross-correlation of the received signal $\vec{r}$ with respect to different satellite signatures.



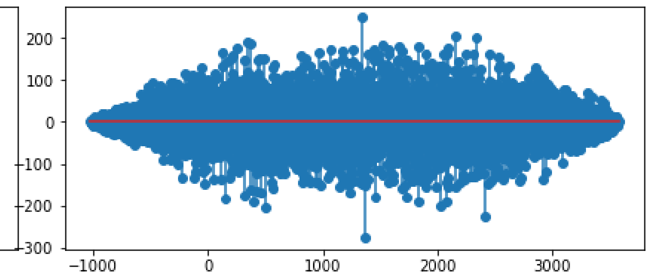
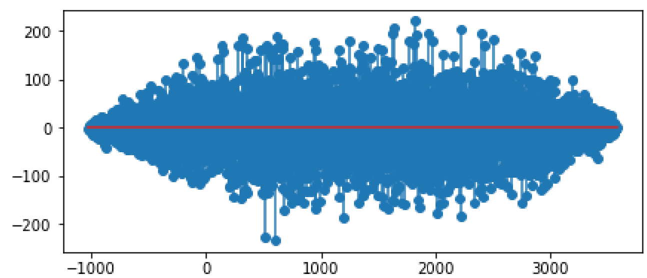


Figure 1: $\text{corr}_{\vec{r}}(\vec{s}_1)[k]$ vs. k

Figure 2: $\text{corr}_{\vec{r}}(\vec{s}_2)[k]$ vs. k



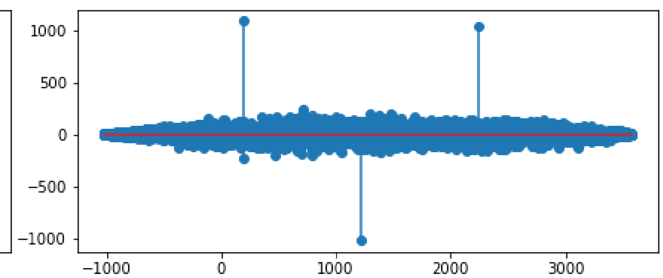


Figure 3: $\text{corr}_{\vec{r}}(\vec{s}_3)[k]$ vs. k

Figure 4: $\text{corr}_{\vec{r}}(\vec{s}_4)[k]$ vs. k

Find out which satellite signals are present and the corresponding transmission delays. Assume a threshold of 800 for determining the peak correlation.

(e) Assume consecutive time samples are $\delta t = 0.1\mu\text{s} = 0.1 \times 10^{-6}\text{s}$ apart. Use your results from the last part to find out distance of satellites from the receiver.

(f) In addition to sending a unique signature signal, a GPS satellite can also "modulate" the signature to communicate more information. Modulating a signature means multiplying the entire signature block by $+1$ or -1.

For example satellite 1 can transmit the sequence $\vec{s_1}$ when it wants send a message bit of 1 or $-\vec{s_1}$ when it wants to send a message bit of -1. Using the correlation plots, find out the message bits the satellites are sending.

