1 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads with up to k different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

Solution:

(a) k^p. For each of the p beads, there are k possibilities for its colors. Therefore, by the first counting principle, there are k^p different sequences.

(b) $k^p - k$. You can have k sequences of a beads with only one color.

(c) Since p is prime, rotating any sequence by less than p spots will produce a new sequence. As in, there is no number x smaller than p such that rotating the beads by x would cause the pattern to look the same. So, every pattern which has more than one color of beads can be rotated to form $p-1$ other patterns. So the total number of patterns equivalent with some bead sequence is p. Thus, the total number of non-equivalent patterns are $(k^p - k)/p$.

(d) $(k^p - k)/p$ must be an integer, because from the previous part, it is the number of ways to count something. Hence, $k^p - k$ has to be divisible by p, i.e., $k^p \equiv k \pmod{p}$, which is Fermat's Little Theorem.

2 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) How many 7-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
i. How many different 13-card bridge hands are there? ii. How many different 13-card bridge hands are there that contain no aces? iii. How many different 13-card bridge hands are there that contain all four aces? iv. How many different 13-card bridge hands are there that contain exactly 6 spades?

(d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(e) How many 99-bit strings are there that contain more ones than zeros?

(f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.
i. How many different anagrams of ALABAMA are there? ii. How many different anagrams of MONTANA are there?

(g) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E (and not necessarily E's neighbor)

(h) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(j) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**

(l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

Solution:

(a) $\binom{n+k}{k}$

(b) There are 3 options for the first digit. For each of the next digits, they only have 2 options because they cannot be equal to the previous digit. Thus, $3 * 2^6$

(c) We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.

We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.

We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.

We need our hand to contain 6 out of the 13 spade cards, and 7 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{6}\binom{39}{7}$ ways to make up the hand.

(d) If we consider the 104! rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number 104! overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52}. Hence, the actual number of rearrangements of 2 identical decks is $104!/2^{52}$.

(e) **Answer 1:** There are $\binom{99}{k}$ 99-bit strings with k ones and $99-k$ zeros. We need $k > 99-k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99}\binom{99}{k}$.
This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99}\binom{99}{k} = \sum_{k=50}^{99}\binom{99}{99-k} = \sum_{l=0}^{49}\binom{99}{l}$$

by substituting $l = 99-k$. Now $\sum_{k=50}^{99}\binom{99}{k} + \sum_{l=0}^{49}\binom{99}{l} = \sum_{m=0}^{99}\binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99}\binom{99}{k} = (1/2)\cdot 2^{99} = 2^{98}$.

Answer 2: Symmetry Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones. Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B. Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B, and since there are 2^{99} 99-bit strings, we get $|A| = |B| = (1/2) \cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

(f) ALABAMA: The number of ways of rearranging 7 distinct letters and is $7!$. In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $4!$ (which is the number of ways of permuting the 4 A's among themselves). Aka, we only want $1/4!$ out of the total rearrangements. Hence, there are $7!/4!$ anagrams.

MONTANA: In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of $2!$ for the number of ways of permuting the 2 A's among themselves and another factor of $2!$ for the number of ways of permuting the 2 N's among themselves). Hence, there are $7!/(2!)^2$ different anagrams.

(g) (1) We consider CE is a new letter X, then the question becomes counting the rearranging of 5 distinct letters, and is $5!$. (2) Symmetry: Let A be the set of all the rearranging of ABCDEF with C on the left side of E, and B be the set of all the rearranging of ABCDEF with C on the right side of E. $|A \bigcup B| = 6!, |A \cap B| = 0$. There is a bijection between A and B by construct a operation of exchange the position of C and E. Thus $|A| = |B| = 6!/2$.

(h) Each ball has a choice of which bin it should go to. So each ball has 27 choices and the 9 balls can make their choices separately. Hence, there are 27^9 ways.

(i) Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k = 9$ and $n = 27$, so there are $\binom{35}{9}$ ways.

(j) **Answer 1:** Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. There are 2 cases to consider:
Case 1: The 2 balls land in the same bin. This gives 7 ways.
Case 2: The 2 balls land in different bins. This gives $\binom{7}{2}$ ways of choosing 2 out of the 7 bins for the balls to land in. Note that it is *not* 7×6 since the balls are identical and so there is no order on them.
Summing up the number of ways from both cases, we get $7 + \binom{7}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. From class (see note 11), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k = 2$ and $n = 7$, we get $\binom{8}{2}$ ways to do this.
EASY EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

(k) **Answer 1:** Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10}(2i-1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are 20! such arrangements. We pair up the students at positions $2i-1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the 20! permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x,y), student x could have appeared in position $2i-1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the 20! permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, 20! overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $20!/(10! \cdot 2^{10})$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2}\binom{18}{2}\dots\binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in 10! ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, ..., tenth step. Hence, we have to divide the above number by 10! to get the number of different pairings. Thus there are $\binom{20}{2}\binom{18}{2}\dots\binom{2}{2}/10!$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

(l) $\binom{n+k}{k}$. This is just n indistinguishable balls into k+1 distinguishable bins (stars and bars). There is a bijection between a sequence of n ones and k plusses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has. Note that this is the exact same answer as part a - make sure you understand why!

(m) $n-1$. It's easy just to enumerate the solutions here. x_0 can take values $1,2,\dots,n-1$ and this uniquely fixes the value of x_1. So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.

(n) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. This is just n-(k+1) indistinguishable balls into distinguishable k+1 bins. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

3 Good Khalil Hunting

As a sidejob, Khalil is also working as a janitor in Berkeley EECS. One day, he notices a problem on the board and decides to solve it. The problem is as follows: **Find all homeomorphically irreducible trees having 10 vertices.** A tree is homeomorphically irreducible if it has no vertices of degree 2. Assume all vertices and edges are indistinguishable from another.

Let's help Khalil solve this using strategic **casework**. We will partition the problem based off the number of the leaves in the tree. For sake of clarity, label the vertices $v_1, \ldots, v_{10}$ and their degrees $d_1, \ldots, d_{10}$ in decreasing order of degree.

(a) Show that the number of leaves, ℓ, we can have is $6 \leq \ell \leq 9$. (*Hint*: What do you know about the degrees of a leaf? What about a non-leaf in this case?)

For the following parts, drawings are neither necessary nor sufficient for your answer, but are highly encouraged to help you get the answer. Please briefly justify your answers by formulating equations involving the degrees of the vertices, along with short explanations.

(b) How many 10 vertex, homeomorphically irreducible trees of 9 leaves are there? Justify your answer.

(c) How many 10 vertex, homeomorphically irreducible trees of 8 leaves are there? Justify your answer.

(d) How many 10 vertex, homeomorphically irreducible trees of 7 leaves are there? Justify your answer.

(e) How many 10 vertex, homeomorphically irreducible trees of 6 leaves are there? Justify your answer.

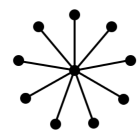
In total, you should have counted 10 trees. Great work!

Solution:

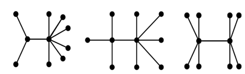
We solve this problem by casework. It is very hard to count upfront the number of 10 vertex homeomorphically irreducible trees. However, counting them with the constraint of how many leaves they contain is much easier. First, we must make sure that our cases are mutually exclusive (no overlap) and collectively exhaustive (covers all the possible cases). They are mutually exclusive because a tree can't have, say, 1 and 2 leaves at the same time. We show in part (a) that as long as we consider the cases of 6,7,8,9 leaves, then that is collectively exhaustive.

(a) We know that the sum of the degrees is 18 as the tree has 9 edges. Note that leaves are degree 1 and non-leaves are at least degree 3 since we aren't considering degree 2 vertices. Thus, if there are k leaves and 10 - k non-leaves, then the sum of the degrees, $18 \leq k + 3 * (10 - k) = 30 - 2k$, thus $k \geq 6$. (the sum of the degrees is lower-bounded when we substitute for all non-leaves $d_i, d_i = 3$). Also, $k \leq 9$ since a 10 vertex tree has at max 9 leaves.

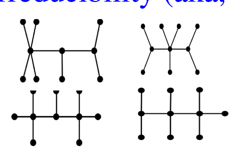
(b) 1. This is a node in the middle with 9 leaves around it.



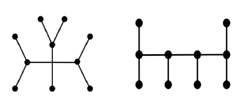
(c) 3. If there are 8 leaves and 2 non-leaves, then the non-leaves, $d_1 + d_2 = 10$ with $d_1, d_2 \geq 3$, so either: $d_1 = 7, d_2 = 3$, or $d_1 = 6, d_2 = 4$, or $d_1 = d_2 = 5$. Each one of these has one tree, because since we are assuming indistinguishability, swapping d_1 and d_2 does not make a difference. This makes up all the cases.



(d) 4. If there are 7 leaves and 3 non-leaves, then the non-leaves, $d_1 + d_2 + d_3 = 11$. Thus the 2 cases are $d_1 = 5, d_2 = 3, d_3 = 3$ or $d_1 = 4, d_2 = 4, d_3 = 3$. This makes up all the cases since any other case will violate $d_i \geq 3$ for non-leaves. Now, for each one of these 2 cases, there are 2 trees. For example, 3,3,5 and 3,5,3 are different trees, however, 5,3,3 is the same as 3,3,5 since they are equivalent by indistinguishability (just flip the tree). Note, there cannot exist any other trees since otherwise we will violate irreducibility (aka, degree 2 nodes will exist).



(e) 2. If there are 6 leaves and 4 non-leaves, then the non-leaves $d_1 + d_2 + d_3 + d_4 = 12$, thus $d_1 = d_2 = d_3 = d_4 = 3$ by pidgeonhole (or just algebra). Sneakily, this case actually has two trees. One of which contains a degree 3 node with no leaves attached to it, while the other does. You can visualize the latter tree containing 4 non-leaves being in a chain as expected, and the former tree containing 4 non-leaves like a perpendicular sign. Note, there cannot exist any other trees since otherwise we will violate irreducibility (aka, degree 2 nodes will exist).



This problem was designed to give extensive practice on casework. If you haven't noticed already, it is based off the problem seen in the movie, Good Will Hunting. The math was heavily derived from this paper: http://math.unideb.hu/media/horvath-gabor/publications/gwh2.pdf The images are derived from this medium article: https://medium.com/cantors-paradise/the-math-problems-from-good-will-hunting-w-solutions-b081895bf379. Congrats, you are now as smart as Matt Damon.

4 August Absurdity

Since March Madness was cancelled, the council unanimously decided to have August Absurdity instead - an online Discrete Mathematics tournament! There are 64 teams (including Cal) in the single-elimination tournament - that means, every match is between two teams and will decide a winner who moves on to the next round and a loser who is eliminated from the tournament. Thus the first round will have 64 teams, the next will have 32, and so on until 1 remains. There is a single, randomly initialized, starting bracket.

(a) How many tournament outcomes exist such that Cal wins the entire tournament?

(b) In the first round, Cal will face a no-name school called LJSU (some people call it Stanfurd?). The format of each match is as follows: Each of the two teams have 8 players labelled from 1 to 8. They play a series of games. In the first game, the two 1's play each other. The loser of the game is eliminated and replaced by the next player of the same team until all players from one team are eliminated, ending the match. What is the number of possible sequences of games such that Cal wins the match?

(c) Cal employs a blasphemous strategy that even baffles themselves. They place their players in an order such that each player is either taller than all the preceding players or shorter than all the preceding players. Let 1-8 represent the players' heights. An example of a valid ordering: 4, 5, 6, 3, 2, 7, 1, 8. An example of an invalid ordering: 1, 2, 3, 4, 5, 6, 8, 7. (invalid since 7 is neither taller or shorter than all the preceding players). How many such orderings exist?

(d) To keep viewership up after the tournament finishes, the council plans an All-Star match. The 16 greatest players in the league were chosen, including Oski and a tree..? Oski refuses to play on the same team as the tree. How many ways can the 16 players be distributed into two teams of 8 players such that Oski and the tree are in opposite teams?

(e) Provide an explanation for the following combinatorial identity. Hint: Solve the previous part using another method. Those two methods should correspond to the two sides of the equality.

$$\binom{n}{r} - \binom{n-2}{r-2} - \binom{n-2}{r} = 2\binom{n-2}{r-1}$$

Solution:

(a) In the first round, there are 32 matches, each with 2 possible outcomes. Thus there are 2^{32} possibilities of the first round. In the next round, there are 16 matches, again each with 2 possible outcomes for a total of 2^{16}. Continuing on this pattern, $2^{32}2^{16}2^{8}2^{4}2^{2}2^{1} = 2^{63}$. More succinctly, the tournament has a total of 63 matches, each with 2 options, thus there are 2^{63} outcomes from the first rule of counting. Now, since we are considering only the outcomes where Cal wins, we have overcounted. We only care about $1/2^6$ of the previous outcomes because we already know that Cal wins in each of the 6 rounds. Thus, the total amount of outcomes is 2^{57}.

(b) Let Cal and Stanford be C and S. We can write the sequence of games as a sequence of letters that represent the winner of each game. For example, SSCCCCCC means Stanford won the first two games and then Cal won the next 8, ending the match. Note that for any sequence of games, there is only 1 way to arrange the unplayed games at the end. So we can append those games to the end and that would not make a difference. If we add the rest of Stanford's remaining letters to the sequence, we are counting the number of 16 letter sequences with 8 C's and 8 S's. So in fact, the total number of sequences of games is $\binom{16}{8}$. (Verify this holds for a small example such as n=4 instead of 16). Now, since we are considering the number of sequences of games such that Cal wins, by symmetry, that is exactly half of these. Thus, the answer is $\binom{16}{8}/2$.

(c) Each element in the set is constrained by everything to the left of it. Notably, the leftmost number can be anything, yet the last number has to be either greater than all the previous elements or smaller than them. We want focus on our most constrained variables since that is easiest to count. So, let's start from the end and work backwards. The last number must be the maximum or minimum element of the set. Thus, it can be only 1 or 8. Now, similarly in either case, the second to last number must be the minimum or maximum element of the remaining set. i.e, if the last element was 8, then the second to last must be 7 or 1. If the last element was 1, then the second to last must be 2 or 8. And so on until the very first element, which only has one remaining option. Thus, by the first rule of counting we get 2^7.

(d) **Answer 1: Complementary** We first count the total number of ways to distribute the players into two teams without any restrictions, which is $\binom{16}{8}$ since we choose 8 to be in team A (and the remaining makes up team B). Then, we subtract off the cases we dont want - that Oski and tree are in the same team. We fix the two onto a team and pick the remaining players to make up team A. The two cases are $\binom{14}{6}$ if both on team A, and $\binom{14}{8}$ if both on team B. Thus, we get $\binom{16}{8}$ - $\binom{14}{6}$ - $\binom{14}{8}$

Answer 2: Casework We count the two cases: Oski A/tree B, and Oski B/tree A. So we fix them on their respective teams, and pick the remaining players to make up team A to get $\binom{14}{7}$ in both cases. Thus, we get $2\binom{14}{7}$

(e) Let n represent the number of players and r represent the number of players on team A. Both methods of the previous subparts are ways of calculating the number of ways to distribute n people into two teams of size r and $n-r$ (in this case, $r = n-r$). LHS is answer 1 and RHS is answer 2. Refer to the previous part to understand the details.

5 School Carpool

(a) n males and n females apply to EECS within UC Berkeley. The EECS department only has n seats available. In how many ways can it admit students? Use the above story for a combinatorial argument to prove the following identity:

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

(b) Among the n admitted students, there is at least one male and at least one female. On the first day, the admitted students decide to carpool to school. The male(s) get in one car, and the female(s) get in another car. Use the above story for a combinatorial argument to prove the following identity:

$$\sum_{k=1}^{n-1} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \cdot \binom{2n-2}{n-2}$$

(Hint: Consider the ways that students are admitted. Also, each car has a driver!)

Solution:

(a) One way of counting is simply $\binom{2n}{n}$, since we must pick n students from $2n$.

The other way is to first pick i males, then $n-i$ females. Equivalently, choose i males to admit, and i females to NOT admit. For a fixed i, this yields $\binom{n}{i}\binom{n}{n-i} = \binom{n}{i}^2$ choices. Thus, over all choices of i:

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

(b) Out of the n males and n females who applied, count the number of ways that accepted students can drive to school.

RHS: First pick one male driver and one female driver from the n male and n female applicants (n^2). Then pick the other $n-2$ accepted students from the pool of $2n-2$ remaining applicants.

LHS: Pick k males and $n-k$ females that were accepted: $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$. Then pick a driver among the k males, and among the $n-k$ females. Because the problem statement says there is at least 1 male and 1 female, k can range from 1 to $n-1$.

6 Flippin' Coins

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$
- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

Solution:

(a) $\Omega = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$

(b) An event must be a subset of Ω, meaning that it must consist of possible outcomes.

- No
- Yes
- Yes
- Yes
- Yes

(c) $\{(T,H,H),(T,H,T),(T,T,H)\}$

(d) $\{(T,H,H),(H,H,T),(H,T,H),(T,T,T)\}$

(e) Since $|\Omega| = 2^3 = 8$ and every outcome has equal probability, $\mathbb{P}[(H,H,T)] = 1/8$.

(f) The event of interest is $E = \{(H,H,T),(H,T,H),(T,H,H)\}$, which has size 3. Whence $\mathbb{P}[E] = 3/8$.

7 Past Probabilified

In this question we review some of the past CS70 topics, and look at them probabilistically.
For the following experiments,

i. Define an appropriate sample space Ω.

ii. Give the probability function $\mathbb{P}(\omega)$.

iii. Compute $\mathbb{P}(E_1)$ given event E_1.

iv. Compute $\mathbb{P}(E_2)$ given event E_2.

(a) Fix a prime $p > 2$, and uniformly sample twice with replacement from $\{0,\dots,p-1\}$ (assume we have two $\{0,\dots,p-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other in $(\bmod p)$ space.
$E_1 =$ The resulting product is 0.
$E_2 =$ The product is $(p-1)/2$.

(b) Make a graph on n vertices by sampling uniformly at random from all possible edges, (assume for each edge we flip a coin and if it is head we include the edge in the graph and otherwise we exclude that edge from the graph).
$E_1 =$ The graph is complete.
$E_2 =$ vertex v_1 has degree d.

Solution:

(a) i. This is essentially the same as throwing two $\{0,\dots,p-1\}$-sided dice, so one appropriate sample space is $\Omega = \{(i,j) \ : \ i,j \in \mathrm{GF}(p)\}$.

ii. Since there are exactly p^2 such pairs, the probability of sampling each one is $\mathbb{P}[(i,j)] = 1/p^2$.

iii. Now in order for the product $i \cdot j$ to be zero, at least one of them has to be zero. There are exactly $2p-1$ such pairs, and so $\mathbb{P}(E_1) = \frac{2p-1}{p^2}$.

iv. For $i \cdot j$ to equal $(p-1)/2$ it doesn't matter what i is as long as $i \neq 0$ and $j \equiv i^{-1}(p-1)/2 \pmod{p}$. Thus $|E_2| = \left|\left\{(i,j) \ : \ j \equiv i^{-1}(p-1)/2\right\}\right| = p-1$, and whence $\mathbb{P}(E_2) = \frac{p-1}{p^2}$.
Alternative Solution for $\mathbb{P}(E_2)$: The previous reasoning showed that $(p-1)/2$ is in no way special, and the probability that $i \cdot j = (p-1)/2$ is the same as $\mathbb{P}(i \cdot j = k)$ for any $k \in \mathrm{GF}(p)$. But $1 = \sum_{k=0}^{p-1} \mathbb{P}(i \cdot j = k) = \mathbb{P}(i \cdot j = 0) + (p-1)\mathbb{P}(i \cdot j = (p-1)/2) = \frac{2p-1}{p^2} + (p-1)\mathbb{P}(i \cdot j = (p-1)/2)$, and so $\mathbb{P}(E_2) = \left(1 - \frac{2p-1}{p^2}\right)/(p-1) = \frac{p-1}{p^2}$ as desired.

(b) i. Since any n-vertex graph can be sampled, Ω is the set of all graphs on n vertices.

ii. As there are $N = 2^{\binom{n}{2}}$ such graphs, the probability of each indivdual one g is $\mathbb{P}(g) = 1/N$ (by the same reasoning that every sequence of fair coin flips is equally likely!).

iii. There is only one complete graph on n vertices, and so $\mathbb{P}(E_1) = 1/N$.

iv. For vertex v_1 to have degree d, exactly d of its $n-1$ possible adjacent edges must be present. There are $\binom{n-1}{d}$ choices for such edges, and for any fixed choice, there are $2^{\binom{n}{2}-(n-1)}$ graphs with this choice. So $\mathbb{P}(E_2) = \frac{\binom{n-1}{d}2^{\binom{n}{2}-(n-1)}}{2^{\binom{n}{2}}} = \binom{n-1}{d}\left(\frac{1}{2}\right)^{n-1}$.

8 Unlikely Events

(a) Toss a fair coin x times. What is the probability that you never get heads?

(b) Roll a fair die x times. What is the probability that you never roll a six?

(c) Suppose your weekly local lottery has a winning chance of $1/10^6$. You buy lottery from them for x weeks in a row. What is the probability that you never win?

(d) How large must x be so that you get a head with probability at least 0.9? Roll a 6 with probability at least 0.9? Win the lottery with probability at least 0.9?

Solution:

(a) 0.5^x.

(b) $(1-1/6)^x$.

(c) $(1-1/10^6)^x$.

(d) (a) For the coin, we want $0.5^x \leq 0.1$, so

$$x \geq \frac{\log 0.1}{\log 0.5} \approx 3.32.$$

(b) For the die, we want: $(5/6)^x \leq 0.1$, so

$$x \geq \frac{\log 0.1}{\log 5/6} \approx 12.6.$$

(c) For the lottery, we want $(1-1/10^6)^x \leq 0.1$, so

$$x \geq \frac{\log 0.1}{\log(1-1/10^6)} \approx 2 \cdot 10^6.$$

(The answer is approximately equal to $(\log 0.1)/(-1/10^6)$, using the approximation for small values $(1-x) \approx e^{-x}$, where $x = 10^{-6}$.)

9 Probability Practice

(a) If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?

(b) A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?

(c) If five numbers are selected at random from the set $\{1,2,3,\ldots,20\}$, what is the probability that their minimum is larger than 5? (A number can be chosen more than once, and the order in which you select the numbers matters)

Solution:

(a) $18!5!/22! = 1/1463$. The 18! comes from 18 "units": 3 physics books, 8 engineering books, 6 biology books and 1 block of math books. The 5! comes from number of ways to arrange the 5 math books within the same block. 22! is just the total number of ways to arrange the books.

(b) $\left(\frac{2^8}{3^8}\right) = 256/6561$. There are 2^8 possible binary-like strings out of 3^8 total possible ternary strings.

(c) $\left(\frac{15^5}{20^5}\right) = 243/1024$. There are 20^5 total possible sequences of numbers we might select. In order for the minimum to be larger than 5, we need to have our entire sequence made up only of the numbers $\{6,7,...,20\}$. Since there are 15 of these numbers, there are 15^5 possible sequences of them.

10 Homework Process and Study Group

You must describe your homework process and study group in order to receive credit for this question.