1 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

(a) There is a real number which is not rational.

(b) All integers are natural numbers or are negative, but not both.

(c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.

(d) $(\forall x \in \mathbb{Z})\ (x \in \mathbb{Q})$

(e) $(\forall x \in \mathbb{Z})\ (((2 \mid x) \lor (3 \mid x)) \implies (6 \mid x))$

(f) $(\forall x \in \mathbb{N})\ ((x > 7) \implies ((\exists a, b \in \mathbb{N})\ (a + b = x)))$

Solution:

(a) $(\exists x \in \mathbb{R})\ (x \notin \mathbb{Q})$, or equivalently $(\exists x \in \mathbb{R})\ \neg(x \in \mathbb{Q})$. This is true, and we can use π as an example to prove it.

(b) $(\forall x \in \mathbb{Z})\ (((x \in \mathbb{N}) \lor (x < 0)) \land \neg((x \in \mathbb{N}) \land (x < 0)))$. This is true, since we define the naturals to contain all integers which are not negative.

(c) $(\forall x \in \mathbb{N})\ ((6 \mid x) \implies ((2 \mid x) \lor (3 \mid x)))$. This is true, since any number divisible by 6 can be written as $6k = (2 \cdot 3)k = 2(3k)$, meaning it must also be divisible by 2.

(d) All real numbers are complex numbers. This is true, since any real number x can equivalently be written as $x + 0i$.

(e) Any integer that is divisible by 2 or 3 is also divisible by 6. This is false–2 provides the easiest counterexample. Note that this statement is false even though its converse (part c) is true.

(f) If a natural number is larger than 7, it can be written as the sum of two other natural numbers. This is trivially true, since we can take $a = x$ and $b = 0$.

(Aside: this is a refererence to the very weak Goldback Conjecture (`https://xkcd.com/1310/`).)

2 Proof Practice

(a) Prove that $\forall n \in \mathbb{N}$, if n is odd, then n^2+1 is even.

(b) Prove that $\forall x, y \in \mathbb{R}$, $\min(x,y) = (x+y-|x-y|)/2$.

(c) Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

(d) Suppose $A \subseteq B$. Prove $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

Solution:

(a) We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k+1$ for some natural number k. Substituting into the expression n^2+1, we get $(2k+1)^2+1$. Simplifying the expression yields $4k^2+4k+2$. This can be rewritten as $2 \times (2k^2+2k+1)$. Since $2k^2+2k+1$ is a natural number, by the definition of even numbers, n^2+1 is even.

(b) We will use a proof by cases. We know the following about the absolute value function for real number z.

$$|z| = \begin{cases} z, & z \geq 0 \\ -z, & z < 0 \end{cases}$$

Case 1: $x < y$. This means $|x-y| = y-x$. Substituting this into the formula on the right hand side, we get

$$\frac{x+y-y+x}{2} = x = \min(x,y).$$

Case 2: $x \geq y$. This means $|x-y| = x-y$. Substituting this into the formula on the right hand side, we get

$$\frac{x+y-x+y}{2} = y = \min(x,y).$$

(c)

$$\sum_{i=1}^{n} i = 1+2+\cdots+n$$

$$2\sum_{i=1}^{n} i = (1+n)+(2+(n-1))+\cdots+(n+1) = (n+1)n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

We can also do a proof by induction.
Base case: for $n = 1$, $\sum i = 1^1 k = 1$.
Inductive hypothesis: Suppose $\sum_{i=1}^{n} k = \frac{n(n+1)}{2}$.

Inductive step: We want to show $\sum_{i=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$. We have

$$\begin{aligned}\sum_{i=1}^{n+1} k &= n+1+\sum_{i=1}^{n} k \\ &= n+1+\frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2}\end{aligned}$$

(d) Suppose $A' \in \mathscr{P}(A)$, that is, $A' \subseteq A$ (by the definition of the power set). We must prove that for any such A', we also have that $A' \in \mathscr{P}(B)$, that is, $A' \subseteq B$.

Let $x \in A'$. Then, since $A' \subseteq A$, $x \in A$. Since $A \subseteq B$, $x \in B$. We have shown $(\forall x \in A')\, x \in B$, so $A' \subseteq B$.

Since the previous argument works for any $A' \subseteq A$, we have proven $(\forall A' \in \mathscr{P}(A))\, A' \in \mathscr{P}(B)$. So, we conclude $\mathscr{P}(A) \subseteq \mathscr{P}(B)$ as desired.

3 Open Set Intersection

For $a,b \in \mathbb{R}$, define an open interval (a,b) as the set $\{x \in \mathbb{R} | x > a \wedge x < b\}$

(a) By this definition, is the empty set an open interval?

(b) Let (a,b) and (c,d) be two open intervals. Prove that $(a,b) \cap (c,d)$ is an open interval

(c) Let $I_1, \dots I_k$ be a finite sequence of open intervals. Prove that for every $k \in \mathbb{N}$, $I_1 \cap I_2 \cap \dots \cap I_k$ is an open interval

(d) Prove that a set containing exactly one number is not an open interval. (**Hint:** You may use the fact that between any two real numbers, there is another real number)

(e) Let $I_1, I_2, \dots$ be an infinite sequence of open intervals. Is it always true that $\cap_{k=1}^{\infty} I_k$ is an open interval? (**hint:** Consider the set $\cap_{k=1}^{\infty}(-1/k, 1/k)$)

(f) why can we not use induction to prove part e?

Solution:

(a) Yes, $(0,0)$ is the empty set. Moreover, if $a \geq b$, then $(a,b) = \emptyset$

(b) We can write $(a,b) \cap (c,d)$ as $\{x \in \mathbb{R} | x > a \wedge x > c \wedge x < b \wedge x < d\} = \{x \in \mathbb{R} | x > \max(a,c) \wedge x < \min(b,d)\} = (\max(a,c), \min(b,d))$. Alternatively, you can prove by cases. By symmetry, we can assume that $x \leq c$. Also we can assume that $a < b$ and $c < d$ so that the two intervals are non-empty.

- case 1: $a \leq b \leq c \leq d$. In this case, $(a,b) \cap (c,d) = \emptyset$
- case 2: $a \leq c \leq b \leq d$. In this case $(a,b) \cap (c,d) = (c,b)$

- case 3: $a \leq c \leq d \leq b$. In this case, $(a,b) \cap (c,d) = (c,d)$

(c) Part b gives the base case: $I_1 \cap I_2$ is an open interval. Our inductive hypothesis is that for some k, $I_1 \cap \ldots I_k$ is an open interval. Our inductive step thus follows from part b, as $(I_1 \cap \ldots I_k) \cap I_{k+1}$ is also an open interval.

(d) Let x be any real number. Suppose, by contradiction, that $\{x\} = (a,b)$ for some $a, b \in \mathbb{R}$. Since $x \in (a,b)$, we must have $a < x < b$. However, there exists a real number, y, such that $a < y < x$, which means $y \neq x$ and $y \in (a,b)$. Thus, (a,b) does not contain just x

(e) Let $K = \cap_{k=1}^{\infty}(-1/k, 1/k)$. We can show that $K = \{0\}$. To show that $0 \in K$, note that for every $k \in \mathbb{N}$, $-1/k < 0 < 1/k$, so $0 \in (-1/k, 1/k)$. To show that for any $x \neq 0$, $x \notin K$, we know that for large enough k, $1/k < |x|$, which means $x \neq (-1/k, 1/k)$ for large enough k. Thus, K has only one element, and by part e, it is not an open interval.

(f) Induction can only be used to prove that a proposition holds for every natural number. However, just because a proposition holds for every finite natural number does not mean that it holds at infinity.

4 Induction

Prove the following using induction:

(a) For all natural numbers $n > 2$, $2^n > 2n + 1$.

(b) For all positive integers n, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(c) For all positive natural numbers n, $\frac{5}{4} \cdot 8^n + 3^{3n-1}$ is divisible by 19.

Solution:

(a) The inequality is true for $n = 3$ because $8 > 7$. Let the inequality be true for $n = k$, such that $2^k > 2k + 1$. Then,

$$2^{k+1} = 2 \cdot 2^k > 2 \cdot (2k+1) = 4k + 2$$

We know $2k > 1$ because k is a positive integer. Thus:

$$4k + 2 = 2k + 2k + 2 > 2k + 1 + 2 = 2k + 3 = 2(k+1) + 1$$

We've shown that $2^{k+1} > 2(k+1) + 1$, which completes the inductive step.

(b) We can verify that the statement is true for $n = 1$. Assume the statement holds for $n = k$, so that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}.$$

Then we can write

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + (k+1)\right) \\
&= (k+1)\left(\frac{2k^2+k+6k+6}{6}\right) \\
&= (k+1)\left(\frac{2k^2+7k+6}{6}\right) \\
&= (k+1)\left(\frac{(2k+3)(k+2)}{6}\right) \\
&= \frac{(k+1)(2(k+1)+1)((k+1)+1)}{6},
\end{aligned}$$

as desired. Since we've shown that the statement holds for $n = k+1$, our proof is complete.

(c) For $n = 1$, the statement is "$10+9$ is divisible by 19", which is true. Assume that the statement holds for $n = k$, such that $\frac{5}{4} \cdot 8^k + 3^{3k-1}$ is divisible by 19. Then,

$$\begin{aligned}
\frac{5}{4} \cdot 8^{k+1} + 3^{3(k+1)-1} &= \frac{5}{4} \cdot 8 \cdot 8^k + 3^{3k+2} \\
&= 8 \cdot \frac{5}{4} \cdot 8^k + 3^3 \cdot 3^{3k-1} \\
&= 8 \cdot \frac{5}{4} \cdot 8^k + 8 \cdot 3^{3k-1} + 19 \cdot 3^{3k-1} \\
&= 8\left(\frac{5}{4} \cdot 8^k + 3^{3k-1}\right) + 19 \cdot 3^{3k-1}
\end{aligned}$$

The first term is divisible by the inductive hypothesis, and the second term is clearly divisible by 19. This completes our proof, as we've shown the statement holds for $k+1$.

5 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{2^n}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.

(b) Try to instead prove the statement $a_n \leq 3^{2^n - 1}$ using induction. Does this statement imply what you tried to prove in the previous part?

Solution:

(a) Try to prove that for every $n \geq 1$, we have $a_n \leq 3^{2^n}$ by induction.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2^1} = 9$.
Inductive Step: For some $n \geq 1$, we assume $a_n \leq 3^{2^n}$. Now, consider $n+1$. We can write:

$$a_{n+1} = 3a_n^2 \leq 3(3^{2^n})^2 = 3 \times 3^{2 \times 2^n} = 3 \times 3^{2^{n+1}} = 3^{2^{n+1}+1}.$$

However, what we wanted was to get an inequality of the form: $a_{n+1} \leq 3^{2^{n+1}}$. There is an extra $+1$ in the exponent of what we derived.

(b) This time the induction works.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2-1} = 3$.
Inductive Step: For some $n \geq 1$ we assume $a_n \leq 3^{2^n-1}$. Now, consider $n+1$. We can write:

$$a_{n+1} = 3a_n^2 \leq 3 \times (3^{2^n-1})^2 = 3 \times 3^{2 \times (2^n-1)} = 3 \times 3^{2^{n+1}-2} = 3^{2^{n+1}-1}.$$

This is exactly the induction hypothesis for $n+1$. Note that for every $n \geq 1$, we have $2^n - 1 \leq 2^n$ and therefore $3^{2^n-1} \leq 3^{2^n}$. This means that our modified hypothesis which we proved here does indeed imply what we wanted to prove in the previous part. This is called "strengthening" the induction hypothesis because we proved a stronger statement and by proving that statement to be true, we proved our original statement to be true as well.

6 A Coin Game

(10 Points) Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3 \cdot 2 = 6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

Solution:

We can prove this by strong induction on n.

Base Case: If $n = 1$, you start with a stack of one coin, so the game immediately terminates. Your total score is zero–and indeed, $\frac{n(n-1)}{2} = \frac{1 \cdot 0}{2} = 0$.

Inductive Step: Suppose that if you start with i coins (for i between 1 and n inclusive), your score will be $\frac{i(i-1)}{2}$ no matter what strategy you employ. Now suppose you start with $n+1$ coins. In your first move, you must split your stack into two smaller stacks. Call the sizes of these stacks s_1 and s_2 (so $s_1+s_2=n+1$ and $s_1,s_2\geq 1$). Your end score comes from three sources: the points you get from making this first split, the points you get from future splits involving coins from stack 1, and the points you get from future splits involving coins from stack 2. From the rules of the game, we know you get s_1s_2 points from the first split. From the inductive hypothesis (which we can apply because s_1 and s_2 are between 1 and n), we know that the total number of points you get from future splits of stack 1 is $\frac{s_1(s_1-1)}{2}$ and similarly that the total number of points you get from future splits of stack 2 is $\frac{s_2(s_2-1)}{2}$, regardless of what strategy you employ in splitting them. Thus, the total number of points we score is

$$\begin{aligned}
s_1s_2+\frac{s_1(s_1-1)}{2}+\frac{s_2(s_2-1)}{2} &= \frac{s_1(s_1-1)+2s_1s_2+s_2(s_2-1)}{2}\\
&= \frac{\big(s_1(s_1-1)+s_1s_2\big)+\big(s_2(s_2-1)+s_1s_2\big)}{2}\\
&= \frac{s_1(s_1+s_2-1)+s_2(s_1+s_2-1)}{2}\\
&= \frac{(s_1+s_2)(s_1+s_2-1)}{2}
\end{aligned}$$

Since $s_1+s_2=n+1$, this works out to $\frac{(n+1)(n+1-1)}{2}$, which is what we wanted to show your total number of points came out to. This completes our proof by induction.

7 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X)=\{y\mid y=f(x) \text{ for some } x\in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y)=\{x\mid f(x)\in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets X and Y, $X=Y$ if and only if $X\subseteq Y$ and $Y\subseteq X$. To prove that $X\subseteq Y$, it is sufficient to show that $(\forall x)\ ((x\in X)\implies(x\in Y))$.

(a) $f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(B)$.

(b) $f(A\cup B)=f(A)\cup f(B)$.

Solution:

In order to prove equality $A=B$, we need to prove that A is a subset of B, $A\subseteq B$ and that B is a subset of A, $B\subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose x is such that $f(x)\in A\cup B$. Then either $f(x)\in A$, in which case $x\in f^{-1}(A)$, or $f(x)\in B$, in which case $x\in f^{-1}(B)$, so in either case we have $x\in f^{-1}(A)\cup f^{-1}(B)$. This proves that $f^{-1}(A\cup B)\subseteq f^{-1}(A)\cup f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Suppose, without loss of generality, that $x \in f^{-1}(A)$. Then $f(x) \in A$, so $f(x) \in A \cup B$, so $x \in f^{-1}(A \cup B)$. The argument for $x \in f^{-1}(B)$ is the same. Hence, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

(b) Suppose that $x \in A \cup B$. Then either $x \in A$, in which case $f(x) \in f(A)$, or $x \in B$, in which case $f(x) \in f(B)$. In either case, $f(x) \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now, suppose that $y \in f(A) \cup f(B)$. Then either $y \in f(A)$ or $y \in f(B)$. In the first case, there is an element $x \in A$ with $f(x) = y$; in the second case, there is an element $x \in B$ with $f(x) = y$. In either case, there is an element $x \in A \cup B$ with $f(x) = y$, which means that $y \in f(A \cup B)$. So $f(A) \cup f(B) \subseteq f(A \cup B)$.

8 Bijective Or Not

For each of the following, determine whether or not it's an injection, subjection, and bijection. Please prove your claims.

(a) $f(x) = 10^{-5}x$

(i) for domain $\mathbb{R}$ and range $\mathbb{R}$

(ii) for domain $\mathbb{Z} \cup \{\pi\}$ and range $\mathbb{R}$

(b) $f(x) = \{x\}$, where the domain is $D = \{0, \ldots, n\}$ and the range is $\mathscr{P}(D)$, the powerset of D (that is, the set of all subsets of D).

(c) Consider the number $X = 1234567890$, and obtain X' by shuffling the order of the digits of X (for example, 2134756890). Is $f(i) = (\mathrm{i}+1)^{\text{st}}$ *digit of* X' a bijection from $\{0, \ldots, 9\}$ to itself?

Solution:

(a) It is bijective for (i), but fails to be surjective in (ii):

(i) Firstly, it is injective because if $f(x) = f(y)$, then $10^{-5}x = 10^{-5}y$ and so multiplying by 10^5 on both sides, we get $x = y$, so no two real numbers can be mapped to the same real numbers. Secondly, it is surjective, because for any $y \in \mathbb{R}$, we have $f(10^5 y) = 10^{-5} \cdot 10^5 y = y$, so each y has an $x = 10^5 y$ that maps to it.

(ii) f is injective for the same reason as above, but it is not a surjection, since the only $x \in \mathbb{R}$ that maps to e.g. 10^{-6} is $x = 10^{-1} \notin \mathbb{Z} \cup \{\pi\}$.

(b) f is injective, but not surjective. There exists a subset $S \subset D$ containing at least two elements of D (in the case of $n = 0$, the two elements are the empty set and 0 itself). However, $f(x)$ always contains exactly one element. Hence there is no $x \in D$ that gets mapped to S.

(c) Yes. Let us show injectivity by noticing that each number between 0 and 9 occurs precisely once in X, and thus precisely once in X' too. As a result, no two digits of X' can be the same. Surjectivity follows from similar reasoning: Since any fixed number $y \in \{0, \ldots, 9\}$ is a digit of X, it must be a digit of X' too, let's call that digit the i_y^{th} digit. Then $f(i_y) = y$.

9 Homework Process and Study Group

You must describe your homework process and study group in order to receive credit for this question.