CS 70 Discrete Mathematics and Probability Theory
Summer 2020 Course Notes **DIS 07B**

1 Confidence Interval Introduction

We observe a random variable X which has mean μ and standard deviation $\sigma \in (0,\infty)$. Assume that the mean μ is unknown, but σ is known.

We would like to give a 95% confidence interval for the unknown mean μ. In other words, we want to give a random interval (a,b) (it is random because it depends on the random observation X) such that the probability that μ lies in (a,b) is at least 95%.

We will use a confidence interval of the form $(X-\varepsilon, X+\varepsilon)$, where $\varepsilon > 0$ is the width of the confidence interval. When ε is smaller, it means that the confidence interval is narrower, i.e., we are giving a more *precise* estimate of μ.

(a) Using Chebyshev's Inequality, calculate an upper bound on $\mathbb{P}\{|X-\mu| \geq \varepsilon\}$.

(b) Explain why $\mathbb{P}\{|X-\mu| < \varepsilon\}$ is the same as $\mathbb{P}\{\mu \in (X-\varepsilon, X+\varepsilon)\}$.

(c) Using the previous two parts, choose the width of the confidence interval ε to be large enough so that $\mathbb{P}\{\mu \in (X-\varepsilon, X+\varepsilon)\}$ is guaranteed to exceed 95%. [Note: Your confidence interval is allowed to depend on X, which is observed, and σ, which is known. Your confidence interval is not allowed to depend on μ, which is unknown.]

Solution:

(a) Since $\mathbb{E}[X] = \mu$ and $\operatorname{Var} X = \sigma^2$, then by Chebyshev's Inequality,

$$\mathbb{P}\{|X-\mu| \geq \varepsilon\} \leq \frac{\operatorname{Var} X}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2}.$$

(b) Note that $|X-\mu| < \varepsilon$ if and only if $-\varepsilon < X-\mu < \varepsilon$, if and only if $\mu-\varepsilon < X < \mu+\varepsilon$. However, the first inequality says that $\mu < X+\varepsilon$ and the second inequality says that $\mu > X-\varepsilon$, that is, $X-\varepsilon < \mu < X+\varepsilon$, which is the same thing as saying $\mu \in (X-\varepsilon, X+\varepsilon)$. So, the events $\{|X-\mu| < \varepsilon\}$ and $\{\mu \in (X-\varepsilon, X+\varepsilon)\}$ are identical.

(c) We want $\mathbb{P}\{\mu \in (X-\varepsilon, X+\varepsilon)\} \geq 0.95$, which is equivalent to

$$\mathbb{P}\{|X-\mu| \geq \varepsilon\} = 1-\mathbb{P}\{|X-\mu| < \varepsilon\} = 1-\mathbb{P}\{\mu \in (X-\varepsilon, X+\varepsilon)\} \leq 0.05.$$

However, we have the bound $\mathbb{P}\{|X-\mu| \geq \varepsilon\} \leq \sigma^2/\varepsilon^2$, so we just need to choose ε big enough so that $\sigma^2/\varepsilon^2 \leq 0.05$. To do this, we want $\varepsilon^2 \geq 20\sigma^2$, or $\varepsilon \geq \sqrt{20}\sigma \approx 4.47\sigma$. Our confidence interval is therefore $(X-4.47\sigma, X+4.47\sigma)$.

2 Poisson Confidence Interval

You collect n samples (n is a positive integer) $X_1, \dots, X_n$, which are i.i.d. and known to be drawn from a Poisson distribution (with unknown mean). However, you have a bound on the mean: from a confidential source, you know that $\lambda \le 2$. Find a $1-\delta$ confidence interval ($\delta \in (0,1)$) for λ using Chebyshev's Inequality. (Hint: a good estimator for λ is the *sample mean* $\bar{X} := n^{-1}\sum_{i=1}^n X_i$)

Solution:

Our estimator for λ is the sample mean $n^{-1}\sum_{i=1}^n X_i$. We apply Chebyshev's Inequality for $\varepsilon > 0$:

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n X_i - \lambda\right| > \varepsilon\right) \le \frac{\mathrm{Var}(n^{-1}\sum_{i=1}^n X_i)}{\varepsilon^2} = \frac{\mathrm{Var}(\sum_{i=1}^n X_i)}{n^2\varepsilon^2} = \frac{\sum_{i=1}^n \mathrm{Var}\,X_i}{n^2\varepsilon^2} = \frac{\mathrm{Var}\,X_1}{n\varepsilon^2} = \frac{\lambda}{n\varepsilon^2}$$
$$\le \frac{2}{n\varepsilon^2}.$$

We want the probability of error to be at most δ, so we set

$$\frac{2}{n\varepsilon^2} \le \delta \implies \varepsilon \ge \sqrt{\frac{2}{n\delta}}.$$

Our $1-\delta$ confidence interval for λ is $(n^{-1}\sum_{i=1}^n X_i - \sqrt{2/(n\delta)}, n^{-1}\sum_{i=1}^n X_i + \sqrt{2/(n\delta)})$.

3 Hypothesis testing

We would like to test the hypothesis claiming that a coin is fair, i.e. $P(H) = P(T) = 0.5$. To do this, we flip the coin $n = 100$ times. Let Y be the number of heads in $n = 100$ flips of the coin. We decide to reject the hypothesis if we observe that the number of heads is less than $50 - c$ or larger than $50 + c$. However, we would like to avoid rejecting the hypothesis if it is true; we want to keep the probability of doing so less than 0.05. Please determine c. (*Hints: use the central limit theorem to estimate the probability of rejecting the hypothesis given it is actually true. Table is provided in the appendix.*)

Solution:

Let X_i be the random variable denoting the result of the i-th flip:

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th flip is head,} \\ 0 & \text{if the } i\text{-th flip is tail.} \end{cases}$$

Then we have $Y = \sum_{i=1}^n X_i$. If the hypothesis is true, then $\mu = \mathbb{E}[X_i] = \frac{1}{2}$ and $\sigma^2 = \mathrm{Var}(X_i) = \frac{1}{2} \cdot \frac{1}{2} =$

$\frac{1}{4}$. By central limit theorem, we know that

$$P\left(\frac{Y - n\mu}{\sqrt{n\sigma^2}} \leq z\right) \approx \Phi(z)$$

$$P\left(\frac{Y - 100 \cdot \frac{1}{2}}{\sqrt{100 \cdot \frac{1}{4}}} \leq z\right) \approx \Phi(z)$$

$$P\left(\frac{Y - 50}{5} \leq z\right) \approx \Phi(z)$$

where

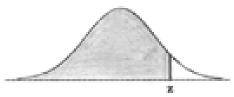
$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathrm{d}x.$$

We will reject the hypothesis when $|Y - 50| > c$. We also want $P(|Y - 50| > c) < 0.05$, or equivalently $P(|Y - 50| \leq c) > 0.95$. We have

$$P(|Y - 50| \leq c) = P\left(\frac{|Y - 50|}{5} \leq \frac{c}{5}\right) \approx 2\Phi(\frac{c}{5}) - 1.$$

The reason this is $\approx 2\Phi(\frac{c}{5}) - 1$ is because the probability we are looking for is the probability that Y is within $\frac{c}{5}$ standard deviations of the mean. By an area argument, we can see that this is $\Phi(\frac{c}{5}) - (1 - \Phi(\frac{c}{5})) = 2\Phi(\frac{c}{5}) - 1$. Let $2\Phi(\frac{c}{5}) - 1 = 0.95$, so $\Phi(\frac{c}{5}) = 0.975$ or $\frac{c}{5} = 1.96$. That is $c = 9.8$ flips. So we see that if we observe more that $50 + 10 = 60$ or less than $50 - 10 = 40$ heads, we can reject the hypothesis.

4 Appendix



Probability Content
from -oo to Z

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Table 1: Table of the Normal Distribution