1 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \le x \le 1, 0 \le y \le 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal distribution of X.

(c) Find the conditional distribution of Y given $X = x$.

(d) Are X and Y independent?

Solution:

(a) Since $f(x,y)$ is a probability density function, it must integrate to 1. Then:

$$1 = \int_0^1 \int_0^2 Cxy\,dy\,dx = \int_0^1 2Cx dx = C$$

Therefore, $C = 1$.

(b) To get the marginal distribution of X, we integrate the joint distribution with respect to Y. So:

$$f_X(x) = \int_0^2 f(x,y)dy = \int_0^2 xydy = 2x$$

This is the marginal distribution for $0 \le x \le 1$.

(c) The conditional distribution of Y given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xy}{2x} = \frac{y}{2}$$

(d) The conditional distribution of Y given $X = x$ does not depend on x, so they are independent.

Alternatively, you could find the marginal distribution of Y and see it is the same as the conditional distribution of Y:

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 xydx = \frac{y}{2}$$

Notice that since X and Y are independent, $f_X(x)f_Y(y) = xy = f_{X,Y}(x,y)$, i.e. the product of the marginal distributions is the same as the joint distribution.

2 Uniform Distribution

You have two spinning wheels, each having a circumference of 10 cm with values in the range $[0,10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinning wheel's mark and Y be the position of the second spinning wheel's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

Solution:

First we write down what we want and expand out the conditioning:

$$\mathbb{P}[X \geq 5 \mid Y \geq X] = \frac{\mathbb{P}[Y \geq X \cap X \geq 5]}{\mathbb{P}[Y \geq X]}.$$

$\mathbb{P}[Y \geq X] = 1/2$ by symmetry. To find $\mathbb{P}[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\mathbb{P}[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

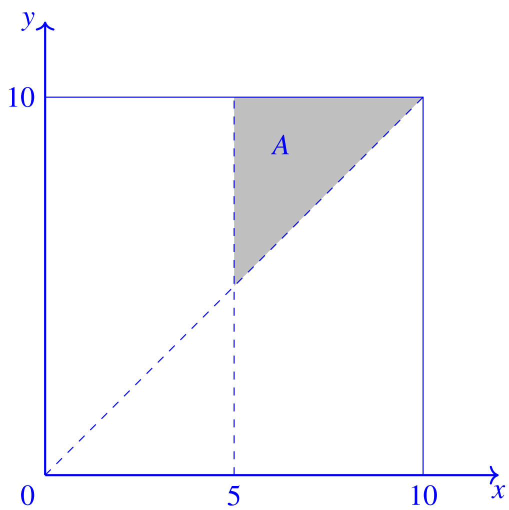

Figure 1: Joint probability density for the spinner.

$$\mathbb{P}[Y \geq X \cap X \geq 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8}.$$

So $\mathbb{P}[X \geq 5 \mid Y \geq X] = (1/8)/(1/2) = 1/4$.

3 Exponential Practice

Let $X \sim \text{Exponential}(\lambda_X)$ and $Y \sim \text{Exponential}(\lambda_Y)$ be independent, where $\lambda_X, \lambda_Y > 0$. Let $U = \min\{X,Y\}$, $V = \max\{X,Y\}$, and $W = V - U$.

(a) Compute $\mathbb{P}(U > t, X \leq Y)$, for $t \geq 0$.

(b) Use the previous part to compute $\mathbb{P}(X \le Y)$. Conclude that the events $\{U > t\}$ and $\{X \le Y\}$ are independent.

(c) Compute $\mathbb{P}(W > t \mid X \le Y)$.

(d) Use the previous part to compute $\mathbb{P}(W > t)$.

(e) Calculate $\mathbb{P}(U > u, W > w)$, for $w > u > 0$. Conclude that U and W are independent. [*Hint*: Think about the approach you used for the previous parts.]

Solution:

(a) One has

$$\begin{aligned}\mathbb{P}(U > t, X \le Y) = \mathbb{P}(t < X \le Y) &= \int_t^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_t^\infty \int_x^\infty \lambda_X \exp(-\lambda_X x)\lambda_Y \exp(-\lambda_Y y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \lambda_X\lambda_Y \int_t^\infty \exp(-\lambda_X x)\cdot\frac{\exp(-\lambda_Y x)}{\lambda_Y}\,\mathrm{d}x = \lambda_X \int_t^\infty \exp\big(-(\lambda_X+\lambda_Y)x\big)\,\mathrm{d}x \\ &= \frac{\lambda_X}{\lambda_X+\lambda_Y}\exp\big(-(\lambda_X+\lambda_Y)t\big).\end{aligned}$$

(b) Take $t = 0$.

$$\mathbb{P}(X \le Y) = \frac{\lambda_X}{\lambda_X+\lambda_Y}.$$

Since X and Y are independent exponentials, $U = \min\{X,Y\} \sim \text{Exponential}(\lambda_X+\lambda_Y)$. So, $\mathbb{P}(U > t) = \exp(-(\lambda_X+\lambda_Y)t)$, and therefore we have $\mathbb{P}(U > t, X \le Y) = \mathbb{P}(X \le Y)\mathbb{P}(U > t)$.

(c) One has

$$\begin{aligned}\mathbb{P}(W > t, X \le Y) = \mathbb{P}(Y - X > t) &= \int_0^\infty \int_{x+t}^\infty \lambda_X \exp(-\lambda_X x)\lambda_Y \exp(-\lambda_Y y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \lambda_X\lambda_Y \int_0^\infty \exp(-\lambda_X x)\cdot\frac{\exp(-\lambda_Y(x+t))}{\lambda_Y}\,\mathrm{d}x \\ &= \lambda_X \exp(-\lambda_Y t)\int_0^\infty \exp\big(-(\lambda_X+\lambda_Y)x\big)\,\mathrm{d}x = \frac{\lambda_X}{\lambda_X+\lambda_Y}\exp(-\lambda_Y t).\end{aligned}$$

So, we see that

$$\mathbb{P}(W > t \mid X \le Y) = \frac{\mathbb{P}(W > t, X \le Y)}{\mathbb{P}(X \le Y)} = \exp(-\lambda_Y t).$$

Alternatively,

$$\begin{aligned}\mathbb{P}(W > t \mid X \le Y) = \mathbb{P}(Y > X + t \mid X \le Y) &= \int_0^\infty \mathbb{P}(Y > x + t \mid Y \ge x) f_X(x)\,\mathrm{d}x \\ &= \exp(-\lambda_Y t)\int_0^\infty f_X(x)\,\mathrm{d}x = \exp(-\lambda_Y t),\end{aligned}$$

where we have used the memoryless property of the exponential distribution. Note that in the first line, we are using conditioning:

$$\mathbb{P}(Y > X + t \mid X \le Y) = \int_0^\infty \mathbb{P}(Y > X + t \mid X \le Y, X = x) f_X(x)\, \mathrm{d}x.$$

The probability inside the integral then becomes $\mathbb{P}(Y > x + t \mid Y \ge x, X = x)$, and then one can drop the conditioning on $X = x$ because X and Y are independent.

(d) By switching X and Y in the previous part, we have

$$\mathbb{P}(W > t \mid Y \le X) = \exp(-\lambda_X t).$$

So, we can use the law of total probability to give

$$\begin{aligned}\mathbb{P}(W > t) &= \mathbb{P}(X \le Y)\mathbb{P}(W > t \mid X \le Y) + \mathbb{P}(Y \le X)\mathbb{P}(W > t \mid Y \le X) \\ &= \frac{\lambda_X}{\lambda_X + \lambda_Y} \exp(-\lambda_Y t) + \frac{\lambda_Y}{\lambda_X + \lambda_Y} \exp(-\lambda_X t).\end{aligned}$$

(e) We calculate

$$\begin{aligned}\mathbb{P}(U > u, W > w, X \le Y) &= \mathbb{P}(u < X \le X + w < Y) \\ &= \int_u^\infty \int_{x+w}^\infty \lambda_X \exp(-\lambda_X x)\lambda_Y \exp(-\lambda_Y y)\, \mathrm{d}y\, \mathrm{d}x \\ &= \lambda_X \lambda_Y \int_u^\infty \exp(-\lambda_X x) \cdot \frac{\exp(-\lambda_Y (x + w))}{\lambda_Y}\, \mathrm{d}x \\ &= \lambda_X \exp(-\lambda_Y w) \int_u^\infty \exp\bigl(-(\lambda_X + \lambda_Y)x\bigr)\, \mathrm{d}x \\ &= \frac{\lambda_X}{\lambda_X + \lambda_Y} \exp(-\lambda_Y w) \exp\bigl(-(\lambda_X + \lambda_Y)u\bigr).\end{aligned}$$

By switching the roles of X and Y in the above computation, we obtain

$$\mathbb{P}(U > u, W > w, Y \le X) = \frac{\lambda_Y}{\lambda_X + \lambda_Y} \exp(-\lambda_X w) \exp\bigl(-(\lambda_X + \lambda_Y)u\bigr).$$

Now, we add the two expressions together to obtain

$$\begin{aligned}\mathbb{P}(U > u, W > w) &= \Bigl(\frac{\lambda_X}{\lambda_X + \lambda_Y} \exp(-\lambda_Y w) + \frac{\lambda_Y}{\lambda_X + \lambda_Y} \exp(-\lambda_X w)\Bigr) \exp\bigl(-(\lambda_X + \lambda_Y)u\bigr) \\ &= \mathbb{P}(W > w)\mathbb{P}(U > u).\end{aligned}$$

So, U and W are independent!