1 Condition on an Event

The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2}, & \text{if } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

Solution:

(a) Integrate:

$$\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = c\int_1^2 x^{-2}\,\mathrm{d}x = -cx^{-1}\Big|_{x=1}^{2} = -c\left(\frac{1}{2}-1\right) = \frac{c}{2} = 1$$

so $c = 2$.

(b) To find $\mathbb{P}(A)$,

$$\mathbb{P}(A) = \int_{1.5}^{2} f_X(x)\,\mathrm{d}x = 2\int_{1.5}^{2} x^{-2}\,\mathrm{d}x = -2x^{-1}\Big|_{x=1.5}^{2} = -2\left(\frac{1}{2}-\frac{2}{3}\right) = \frac{1}{3}.$$

The conditional PDF is thus

$$f_{X|A}(x) = \frac{f_X(x)}{\mathbb{P}(A)} = 6x^{-2}, \qquad x \in [1.5, 2].$$

2 Max of Uniforms

Let $X_1, \ldots X_n$ be independent $U[0,1]$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of n.

(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

Solution:

(a) $Pr[X \leq x] = x^n$ since in order for $\max(X_1, ...X_n) < x$, we must have $X_i < x$ for all i. Since they are independent, we can multiply together the probabilities of each of them being less than x, which is x itself, as their distributions are uniform.

(b) Taking the derivative of the cdf, we have $f_X(x) = nx^{n-1}$

(c)

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^1 x f_X(x) \\
&= \int_0^1 nx^n dx \\
&= \frac{n}{n+1}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_0^1 x^2 f_X(x) = \int_0^1 nx^{n+1} dx = \frac{n}{n+2} \\
Var[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{n}{n+2} - \frac{n^2}{(n+1)^2}
\end{aligned}$$

3 Darts but with ML

Suppose Alice and Bob are playing darts on a circular board with radius 1. When Alice throws a dart, the distance of the dart from the center is uniform $[0, 1]$. When Bob throws the dart, the location of the dart is uniform over the whole board. Let X the a random variable corresponding to the distance of the player's dart from the board.

(a) What is the pdf of X if Alice throws

(b) What is the pdf of X if Bob throws

(c) Suppose we let Alice throw the dart with probability p, and let Bob throw otherwise. What is the pdf of X (your answer should be in terms of p)?

(d) Using the same premise as in part c, suppose you observe a dart on the board but don't know who threw it. Let x be the dart's distance from the center. We would like to come up with a decision rule to determine whether Alice or Bob is more likely to have thrown the dart given your observation, x. Specifically, if we let A be the event that Alice threw the dart and B be the event that Bob threw, we want to guess A if $\mathbb{P}[A|X \in [x, x+dx]] > \mathbb{P}[B|X \in [x, x+dx]]$ (what do these two probabilities have to sum up to?). For what values of x would we guess A? (your answer should be in terms of p)

Solution:

(a) If Alice threw, then $X \sim U[0,1]$, so it's pdf is $f_{X|A}(x|A) = 1$. Note, the cdf is $\mathbb{P}[X < x|A] = \int_0^x 1dx = x$, which makes sense because this is exactly the area of a rectangle of length x and height 1.

(b) If Bob throws, then the probability that $X < x$ is equaled to the area of the disc of radius x around the center of the dartboard divided by the area of the dartboard. Thus, we have the cdf as:

$$\begin{aligned} \mathbb{P}[X < x|B] &= \frac{\pi x^2}{\pi} = x^2 \\ f_{X|B}(x|B) &= \frac{d}{dx}\mathbb{P}[X < x|B] = 2x \end{aligned}$$

(c) To find the pdf if X, we can again take the cdf first and take the derivative:

$$\begin{aligned} \mathbb{P}[X < x] &= \mathbb{P}[X < x|A]\mathbb{P}[A] + \mathbb{P}[X < x|B]\mathbb{P}[B] \\ &= px + (1-p)x^2 \\ f_X(x) &= p + 2(1-p)x \end{aligned}$$

(d) Intuitively, we can sketch out the pdfs of both Alice and Bob's throws and we see that Alice is more likely to hit closer to center compared to Bob. Thus it makes sense to say that there is a particular value x^* such that the distance of the dart from the center is less than x^*, then we guess Alice. Otherwise, we guess Bob. Specifically, we can compute with Bayes's rule:

$$\begin{aligned} \mathbb{P}[A|X \in [x, x+dx]] &= \frac{\mathbb{P}[X \in [x, x+dx]|A]\mathbb{P}[A]}{\mathbb{P}[X \in [x, x+dx]]} \\ &= \frac{f_{X|A}(x|A)dx * \mathbb{P}[A]}{f_X(X) * dx} \\ &= \frac{p}{p + 2(1-p)x} \end{aligned}$$

Note that this function is monotonically decreasing in x. In particular, we want to guess Alice if it is more likely that she threw the dart than Bob threw the dart, which means $\mathbb{P}[A|X \in [x, x+dx]] > 1/2$. Thus, we guess Alice if:

$$\begin{aligned} \frac{p}{p + 2(1-p)x} &> \frac{1}{2} \\ 2x(1-p) &< 2p - p \\ x &< \frac{p}{2(1-p)} \end{aligned}$$

Note that if we take $p = 1/2$ and plot out the conditional pdfs of Alice and Bob, we see that Alice's pdf is higher when $x < 1/2$ and Bob's pdf is higher when $x > 1/2$. Incidentally, if we take $p = 1/2$, we see that our decision boundary is exactly 1/2. Thus, the decision boundary corresponds to the point where the two pdfs, after scaling by p and $1 - p$, have the same height.