1 Warm-up

For each of the following parts, you may leave your answer as an expression.

(a) You throw darts at a board until you hit the center area. Assume that the throws are i.i.d. and the probability of hitting the center area is $p = 0.17$. What is the probability that you hit the center on your eighth throw?

(b) Let $X \sim \text{Geometric}(0.2)$. Calculate the expectation and variance of X.

(c) Suppose the accidents occurring weekly on a particular stretch of a highway is Poisson distributed with average number of accidents equal to 3 cars per week. Calculate the probability that there is at least one accident this week.

(d) Consider an experiment that consists of counting the number of α particles given off in a one-second interval by one gram of radioactive material. If we know from past experience that, on average, 3.2 such α-particles are given off per second, what is a good approximation to the probability that no more that 2 α-particles will appear in a second?

Solution:

(a) $(0.17)(1-0.17)^7$

Let N denote the random variable that you hit the center on your X-th turn. Then $X \sim \text{Geometric}(0.17)$ and hence,

$$\mathbb{P}(X = 8) = (0.17)(1-0.17)^7 \approx 0.0461.$$

(b) $\mathbb{E}(X) = 5$ **and** $\text{Var}(X) = 20$

This follows from $\mathbb{E}(X) = 1/p$ and $\text{Var}(X) = (1-p)/(p^2)$ for $X \sim \text{Geometric}(p)$ as seen in lecture.

(c) $1 - e^{-3}$

Let X denote the number of accidents occurring on the stretch of highway in question during this week. We have $X \sim \text{Poisson}(3)$ and hence,

$$\begin{aligned}\mathbb{P}(X \geq 1) &= 1 - \mathbb{P}(X = 0), \\ &= 1 - e^{-3}\frac{3^0}{0!} \\ &= 1 - e^{-3} \approx 0.9502.\end{aligned}$$

(d) $e^{-3.2} + 3.2e^{-3.2} + \frac{(3.2)^2}{2}e^{-3.2}$

We model the number of α-particles given off during the second considered as a Poisson random variable with parameter $\lambda = 3.2$. Hence,

$$\mathbb{P}(X \leq 2) = e^{-3.2} + 3.2e^{-3.2} + \frac{(3.2)^2}{2}e^{-3.2} = 0.382.$$

2 Coupon Collector Variance

It's that time of the year again - Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\text{Var}(X) = n^2\left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}(X)$. *[Hint: Try to express the number of visits as a sum of geometric random variables as with the coupon collector's problem. Are the variables independent?]*

Solution:

Note that this is the coupon collector's problem, but now we have to find the variance. Let X_i be the number of visits we need to make before we have collected the ith unique Monopoly card actually obtained, given that we have already collected $i-1$ unique Monopoly cards. Then $X = \sum_{i=1}^{n} X_i$ and each X_i is geometrically distributed with $p = (n-i+1)/n$. Moreover, the X_i's themselves are independent, since each time you collect a new card, you are starting from a clean slate.

$$\begin{aligned}
\mathrm{Var}(X) &= \sum_{i=1}^{n} \mathrm{Var}(X_i) && \text{(as the } X_i \text{ are independent)} \\
&= \sum_{i=1}^{n} \frac{1-(n-i+1)/n}{[(n-i+1)/n]^2} && \text{(variance of a geometric r.v. is } (1-p)/p^2\text{)} \\
&= \sum_{j=1}^{n} \frac{1-j/n}{(j/n)^2} && \text{(by noticing that } n-i+1 \text{ takes on all values from 1 to } n\text{)} \\
&= \sum_{j=1}^{n} \frac{n(n-j)}{j^2} \\
&= \sum_{j=1}^{n} \frac{n^2}{j^2} - \sum_{j=1}^{n} \frac{n}{j} \\
&= n^2 \left(\sum_{j=1}^{n} \frac{1}{j^2} \right) - \mathbb{E}(X) && \text{(using the coupon collector problem expected value).}
\end{aligned}$$

3 Boutique Store

Consider a boutique store in a busy shopping mall. Every hour, a large number of people visit the mall, and each independently enters the boutique store with some small probability. The store owner decides to model X, the number of customers that enter her store during a particular hour, as a Poisson random variable with mean λ.

Suppose that whenever a customer enters the boutique store, they leave the shop without buying anything with probability p. Assume that customers act independently, i.e. you can assume that they each flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as Y and the number of them that do not buy anything as Z (so $X = Y + Z$).

(a) What is the probability that $Y = k$ for a given k? How about $\mathbb{P}[Z = k]$? *Hint*: You can use the identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

(b) State the name and parameters of the distribution of Y and Z.

(c) Prove that Y and Z are independent. In particular, prove that for every pair of values y, z, we have $\mathbb{P}[Y = y, Z = z] = \mathbb{P}[Y = y]\mathbb{P}[Z = z]$.

Solution:

(a) We consider all possible ways that the event $Y = k$ might happen: namely, $k + j$ people enter the store ($X = k + j$) and then exactly k of them choose to buy something. That is,

$$\begin{aligned}
\mathbb{P}[Y = k] &= \sum_{j=0}^{\infty} \mathbb{P}[X = k + j] \cdot \mathbb{P}[Y = k \mid X = k + j] \\
&= \sum_{j=0}^{\infty} \left(\frac{\lambda^{k+j}}{(k+j)!} \mathrm{e}^{-\lambda} \right) \cdot \left(\binom{k+j}{k} p^j (1-p)^k \right) \\
&= \sum_{j=0}^{\infty} \frac{\lambda^{k+j}}{(k+j)!} \mathrm{e}^{-\lambda} \cdot \frac{(k+j)!}{k!j!} p^j (1-p)^k \\
&= \frac{(\lambda(1-p))^k \mathrm{e}^{-\lambda}}{k!} \cdot \sum_{j=0}^{\infty} \frac{(\lambda p)^j}{j!} \\
&= \frac{(\lambda(1-p))^k \mathrm{e}^{-\lambda}}{k!} \cdot \mathrm{e}^{\lambda p} \\
&= \frac{(\lambda(1-p))^k \mathrm{e}^{-\lambda(1-p)}}{k!}.
\end{aligned}$$

The case for Z is completely analogous:

$$\mathbb{P}[Z = k] = \frac{(\lambda p)^k \mathrm{e}^{-\lambda p}}{k!}$$

(b) Y follows the Poisson distribution with parameter $\lambda(1-p)$ and Z follows the Poisson distribution with parameter λp.

(c) The joint distribution of Y and Z is given by

$$\begin{aligned}
\mathbb{P}(Y = y, Z = z) &= \sum_{x=0}^{\infty} \mathbb{P}(X = x, Y = y, Z = z) \\
&= \sum_{x=0}^{\infty} \mathbb{P}(Y = y, Z = z \mid X = x) \mathbb{P}(X = x) \\
&= \mathbb{P}(Y = y, Z = z \mid X = y + z) \mathbb{P}(X = y + z) \\
&= \frac{\cancel{(y+z)!}}{y!z!} p^z (1-p)^y \frac{\mathrm{e}^{-\lambda} \lambda^{y+z}}{\cancel{(y+z)!}} \\
&= \frac{\mathrm{e}^{-\lambda(1-p)} (\lambda(1-p))^y}{y!} \cdot \frac{\mathrm{e}^{-\lambda p} (\lambda p)^z}{z!} \\
&= \mathbb{P}(Y = y) \cdot \mathbb{P}(Z = z).
\end{aligned}$$

Since $\mathbb{P}(Y = y, Z = z) = \mathbb{P}(Y = y) \cdot \mathbb{P}(Z = z)$ for all $y, z \in \mathbb{N}$, we get that Y and Z are independent.