1 Let's Talk Probability

(a) When is $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ true? What is the general expression for $\mathbb{P}(A \cup B)$ that is always true.

(b) When is $\mathbb{P}(A \cap B) = \mathbb{P}(A) * \mathbb{P}(B)$ true? What is the general expression for $\mathbb{P}(A \cap B)$ that is always true.

(c) If A and B are disjoint, does that imply they're independent?

Solution:

(a) In general, we know $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$. This is the Inclusion-Exclusion Principle. Therefore if A and B are disjoint, such that $\mathbb{P}(A \cap B) = 0$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ holds.

(b) $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ holds if and only if A and B are independent (by definition). The general rule that always holds is $\mathbb{P}(A \cap B) = \mathbb{P}(B|A)\mathbb{P}(A) = \mathbb{P}(A|B)\mathbb{P}(B)$.

(c) No, if two events are disjoint, we cannot conclude they are independent. Consider a roll of a fair six-sided die. Let A be the event that we roll a 1, and let B be the event that we roll a 2. Certainly A and B are disjoint, as $\mathbb{P}(A \cap B) = 0$. But these events are not independent: $\mathbb{P}(B \mid A) = 0$, but $\mathbb{P}(B) = 1/6$.

Since disjoint events have $\mathbb{P}(A \cap B) = 0$, we can see that the only time when disjoint A and B are independent is when either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

2 Balls and Bins

Throw n balls into n labeled bins one at a time.

(a) What is the probability that the first bin is empty?

(b) What is the probability that the first k bins are empty?

(c) Let A be the event that at least k bins are empty. Notice that there are $m = \binom{n}{k}$ sets of k bins out of the total n bins. If we assune A_i is the event that the i^{th} set of k bins is empty. Then we can write A as the union of A_i's.

$$A = \bigcup_{i=1}^{m} A_i.$$

Write the union bound for the probability A.

(d) Use the union bound to give an upper bound on the probability A from part (c).

(e) What is the probability that the second bin is empty given that the first one is empty?

(f) Are the events that "the first bin is empty" and "the first two bins are empty" independent?

(g) Are the events that "the first bin is empty" and "the second bin is empty" independent?

Solution: Since the balls are thrown one at a time, there is an ordering, and so we are sampling with replacement where order matters rather than where it doesn't (which would correspond to each configuration in the stars and bars setup being equally likely).

(a) The probability that ball i does not land in the first bin is $\frac{n-1}{n}$. The probability that all of the balls do not land in the first bin is $\left(\frac{n-1}{n}\right)^n$.

(b) The probability that ball i does not land in the first k bins is $\frac{n-k}{n}$. The probability that all of the balls do not land in the first k bins is $\left(\frac{n-k}{n}\right)^n$.

(c) We use the union bound. Then

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} \mathbb{P}(A_i)$$

(d) We know the probability of the first k bins being empty from part (b), and this is true for any set of k bins, so

$$\mathbb{P}(A_i) = \left(\frac{n-k}{n}\right)^n.$$

Then,

$$\mathbb{P}(A) \leq m \cdot \left(\frac{n-k}{n}\right)^n = \binom{n}{k}\left(\frac{n-k}{n}\right)^n.$$

(e) Using Bayes' Rule:

$$\begin{aligned}\mathbb{P}[\text{2nd bin empty} \mid \text{1st bin empty}] &= \frac{\mathbb{P}[\text{2nd bin empty} \cap \text{1st bin empty}]}{\mathbb{P}[\text{1st bin empty}]} \\ &= \frac{(n-2)^n/n^n}{(n-1)^n/n^n} \\ &= \left(\frac{n-2}{n-1}\right)^n\end{aligned} \tag{1}$$

Alternate solution:
We know bin 1 is empty, so each ball that we throw can land in one of the remaining $n-1$

bins. We want the probability that bin 2 is empty, which means that each ball cannot land in bin 2 either, leaving $n-2$ bins. Thus for each ball, the probability that bin 2 is empty given that bin 1 is empty is $(n-2)/(n-1)$. For n total balls, this probability is $[(n-2)/(n-1)]^n$.

(f) They are dependent. Knowing the latter means the former happens with probability 1.

(g) In part (c) we calculated the probability that the second bin is empty given that the first bin is empty: $[(n-2)/(n-1)]^n$. The probability that the second bin is empty (without any prior information) is $[(n-1)/n]^n$. Since these probabilities are not equal, the events are dependent.

3 Pairs of Beads

Sinho has a set of $2n$ beads ($n \geq 2$) of n different colors, such that there are two beads of each color. He wants to give out pairs of beads as gifts to all the other $n-1$ TAs, and plans on keeping the final pair for himself (since he is, after all, also a TA). To do so, he first chooses two beads at random to give to the first TA he sees. Then he chooses two beads at random from those remaining to give to the second TA he sees. He continues giving each TA he sees two beads chosen at random from his remaining beads until he has seen all $n-1$ TAs, leaving him with just the two beads he plans to keep for himself. Prove that the probability that at least one of the other TAs (*not* including Sinho himself) gets two beads of the same color is at most $\frac{1}{2}$.

Solution:

We first examine the probability that any given TA gets two beads of the same color given no information about what beads any other TA got. Since we have no information about what anyone else got, we can do our calculations as if our arbitrarily chosen TA was actually the first TA Sinho gave beads to. In this case, no matter what the first bead Sinho chose to give this TA was, for the second bead Sinho has $2n-1$ choices of which only 1 results in the other TA getting two beads of the same color. Thus, the probability that a given TA gets two beads of the same color is $\frac{1}{2n-1}$.

Of course, this is not immediately the probability we are interested in–we actually want to know the probability that *any* of the other TAs gets two beads of the same color. However, we notice that the event we're interested in is just the union over all $n-1$ TAs of the event we already calculated the probability for. Thus, we can apply a union bound, which tells us that the probability we're looking for is no bigger than $(n-1)\frac{1}{2n-1} \leq \frac{n-1}{2n-2} = \frac{1}{2}$, which was the bound we were looking for.