CS 70 Discrete Mathematics and Probability Theory
Summer 2020 Course Notes **Dis 05A**

1 Rain and Wind

The local weather channel just released a statistic for the months of November and December. It said that the probability that it would rain on a windy day is 0.3 and the probability that it would rain on a non-windy day is 0.8. The probability of a day being windy is 0.2. As a student in EECS 70, you are curious to play around with these numbers. Find the probability that:

(a) A given day is both windy and rainy.

(b) A given day is rainy.

(c) For a given pair of days, exactly one of the two days is rainy. (You may assume that the weather on the first day does not affect the weather on the second.)

Solution:

(a) Let R be the event that it rains on a given day and W be the event that a given day is windy. We are given $\mathbb{P}(R \mid W) = 0.3$, $\mathbb{P}(R \mid \overline{W}) = 0.8$ and $\mathbb{P}(W) = 0.2$. Then probability that a given day is both rainy and windy is $\mathbb{P}(R \cap W) = \mathbb{P}(R \mid W)\mathbb{P}(W) = 0.3 \times 0.2 = 0.06$.

(b) Probability that it rains on a given day is $\mathbb{P}(R) = \mathbb{P}(R \mid W)\mathbb{P}(W) + \mathbb{P}(R \mid \overline{W})\mathbb{P}(\overline{W}) = 0.3 \times 0.2 + 0.8 \times 0.8 = 0.7$.

(c) Let R_1 and R_2 be the events that it rained on day 1 and day 2 respectively. Since the weather on the first day doesn't affect that of the second, $\mathbb{P}(R_1) = \mathbb{P}(R_2) = \mathbb{P}(R)$. The required probability is then just $\mathbb{P}(R_1 \cap \overline{R_2}) + \mathbb{P}(\overline{R_1} \cap R_2) = \mathbb{P}(R_1)\mathbb{P}(\overline{R_2}) + \mathbb{P}(\overline{R_1})\mathbb{P}(R_2) = 2 \cdot 0.7 \cdot 0.3 = 0.42$. Since the weather on the first day does not affect the weather on the second day we can multiply the probabilities.

2 Poisoned Smarties

Supposed there are 3 men who are all owners of their own Smarties factories. Burr Kelly, being the brightest and most innovative of the men, produces considerably more Smarties than his competitors and has a commanding 45% of the market share. Yousef See, who inherited his riches, lags behind Burr and produces 35% of the world's Smarties. Finally Stan Furd, brings up the rear with a measly 20%. However, a recent string of Smarties related food poisoning has forced the FDA investigate these factories to find the root of the problem. Through his investigations, the inspector found that one Smarty out of every 100 at Kelly's factory was poisonous. At See's factory, 1.5% of

Smarties produced were poisonous. And at Furd's factory, the probability a Smarty was poisonous was 0.02.

(a) What is the probability that a randomly selected Smarty will be safe to eat?

(b) If we know that a certain Smarty didn't come from Burr Kelly's factory, what is the probability that this Smarty is poisonous?

(c) Given this information, if a randomly selected Smarty is poisonous, what is the probability it came from Stan Furd's Smarties Factory?

Solution:

(a) Let S be the event that a smarty is safe to eat.
Let BK be the event that a smarty is from Burr Kelly's factory.
Let YS be the event that a smarty is from Yousef See's factory.
Finally, let SF be the event that a smarty is from Stan Furd's factory.

$$\begin{aligned}\mathbb{P}(S) &= \mathbb{P}(BK)P(S \mid BK) + \mathbb{P}(YS)\mathbb{P}(S \mid YS) + \mathbb{P}(SF)\mathbb{P}(S \mid SF) \\ &= (0.45)(0.99) + (0.35)(0.985) + (0.2)(0.98) = 0.98625.\end{aligned}$$

Therefore the probability that a Smarty is safe to eat is about 0.98625.

(b) Let P be the event that a smarty is poisonous.

$$\begin{aligned}\mathbb{P}(P \mid \neg BK) &= \frac{\mathbb{P}(\neg BK \cap P)}{\mathbb{P}(\neg BK)} \\ &= \frac{\mathbb{P}(YS \cap P)}{\mathbb{P}(\neg BK)} + \frac{\mathbb{P}(SF \cap P)}{\mathbb{P}(\neg BK)} \quad [\because BK, YS, SF \text{ are mutually exclusive, collectively exhaustive}] \\ &= \frac{\mathbb{P}(YS)}{\mathbb{P}(\neg BK)}\mathbb{P}(P \mid YS) + \frac{\mathbb{P}(SF)}{\mathbb{P}(\neg BK)}\mathbb{P}(P \mid SF) \\ &= \frac{0.35}{0.55} \cdot 0.015 + \frac{0.2}{0.55} \cdot 0.02 = 0.0168.\end{aligned}$$

(c) From Bayes' Rule, we know that:

$$\mathbb{P}(SF \mid P) = \frac{\mathbb{P}(P \mid SF)\mathbb{P}(SF)}{\mathbb{P}(P)}$$

In the first part we calculate the probability that any random Smarty was safe to eat. We can use that since $\mathbb{P}(P) = 1 - \mathbb{P}(S)$. Therefore the solution becomes:

$$\begin{aligned}\mathbb{P}(SF \mid P) &= \frac{\mathbb{P}(P \mid SF)\mathbb{P}(SF)}{1 - \mathbb{P}(S)} \\ &= \frac{(0.02)(0.2)}{(1 - 0.98625)} = 0.29.\end{aligned}$$

3 Bag of Coins

Your friend Forrest has a bag of n coins. You know that k are biased with probability p (i.e. these coins have probability p of being heads). Let F be the event that Forrest picks a fair coin, and let B be the event that Forrest picks a biased coin. Forrest draws three coins from the bag, but he does not know which are biased and which are fair.

(a) What is the probability of three coins being pulled in the order FFB?

(b) What is the probability that the third coin he draws is biased?

(c) What is the probability of picking at least two fair coins?

(d) Given that Forrest flips the second coin and sees heads, what is the probability that this coin is biased?

Solution:

(a) The probability of picking F for the first coin is $(n-k)/n$. The probability of picking F for the second coin, after picking one fair coin already is $(n-k-1)/(n-1)$. The probability of picking B for the third coin is $k/(n-2)$. Thus, the probability of picking the exact sequence FFB is

$$\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}.$$

(b) One approach is to condition on the possible outcomes for the first and second coins

$$\{FF, FB, BF, BB\}$$

such that

$$\mathbb{P}(T) = \mathbb{P}(T \cap FF) + \mathbb{P}(T \cap FB) + \mathbb{P}(T \cap BF) + \mathbb{P}(T \cap BB)$$

where T is the event that the third coin is biased.

A simpler approach is to use the notion of symmetry. We can envision this by laying out all the coins in a line. If you pick the third coin in the line, the probability that coin is biased is the same as the probability that the first coin in the line is biased, or second, or tenth, etc. If we have no information about any other coins, the probability that any single coin is biased is still k/n.

(c) Note that the probability of picking any sequence of two fair coins and a biased coin is the same. It is in fact the probability from part (a). We need to multiply by the number of arrangements of biased and fair coins, however. So, the probability of picking any sequence with two fair coins is

$$\binom{3}{1}\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}.$$

We additionally need to consider the probability of getting 3 fair coins.

$$\frac{(n-k)!(n-3)!}{n!(n-k-3)!}$$

We simply sum the two to get our answer:

$$\binom{3}{1}\frac{(n-k)(n-k-1)k}{n(n-1)(n-2)}+\frac{(n-k)!(n-3)!}{n!(n-k-3)!}$$

(d) We can apply Bayes Rule. Let H denote the event that Forrest sees heads.

$$\mathbb{P}(B \mid H)=\frac{\mathbb{P}(H \mid B)\mathbb{P}(B)}{\mathbb{P}(H)}$$

Note that $\mathbb{P}(H \mid B)=p$ and that $\mathbb{P}(B)=k/n$. We can now compute the denominator. Using the law of total probability, we can expand $\mathbb{P}(H)$.

$$\begin{aligned}\mathbb{P}(H) &= \mathbb{P}(H \mid B)\mathbb{P}(B)+\mathbb{P}(H \mid F)\mathbb{P}(F) \\ &= p\frac{k}{n}+\frac{1}{2}\frac{n-k}{n} \\ &= \frac{2pk+n-k}{2n}\end{aligned}$$

We now combine both parts to get our answer:

$$\frac{p\cdot(k/n)}{(2pk+n-k)/(2n)}=\frac{2pk}{2pk+n-k}.$$