1 Sample Space and Events

Consider the sample space Ω of all outcomes from flipping a coin 3 times.

(a) List all the outcomes in Ω. How many are there?

(b) Let A be the event that the first flip is a heads. List all the outcomes in A. How many are there?

(c) Let B be the event that the third flip is a heads. List all the outcomes in B. How many are there?

(d) Let C be the event that the first and third flip are heads. List all outcomes in C. How many are there?

(e) Let D be the event that the first or the third flip is heads. List all outcomes in D. How many are there?

(f) Are the events A and B disjoint? Express C in terms of A and B. Express D in terms of A and B.

(g) Suppose now the coin is flipped $n \geq 3$ times instead of 3 flips. Compute $|\Omega|, |A|, |B|, |C|, |D|$.

(h) Your gambling buddy found a website online where he could buy trick coins that are heads or tails on both sides. He puts three coins into a bag: one coin that is heads on both sides, one coin that is tails on both sides, and one that is heads on one side and tails on the other side. You shake the bag, draw out a coin at random, put it on the table without looking at it, then look at the side that is showing. Suppose you notice that the side that is showing is heads. What is the probability that the other side is heads? Show your work. [*Hint*: The answer is NOT $1/2$.]

Solution:

(a) Each flip results in either heads (H) or tails (T). So in total the total number of outcomes is 8, which we represent by length 3 strings of H's and T's. We have

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

(b) These are the strings that start with H. We have $A = \{HHH, HHT, HTH, HTT\}$. There are 4 such outcomes.

(c) These are the strings that end with an H. We have $B = \{HHH, HTH, THH, TTH\}$. There are 4 such outcomes.

(d) These are the strings that start and end with an H. We have $C = \{HHH, HTH\}$. There are 2 such outcomes.

(e) We have $D = \{HHH, HHT, HTH, HTT, THH, TTH\}$. There are 6 such outcomes.

(f) No, A and B are not disjoint. For example HHH belongs to both of them.

The event C is the intersection of A and B, because in C we require exactly both A (the first coin being heads) and B (the third coin being heads) to happen. So $C = A \cap B$.

The event D is the union of A and B, because in D we require at least one of A (the first coin being heads) or B (the second coin being heads) to happen. So $D = A \cup B$.

(g) First, obviously $|\Omega| = 2^n$.

Note that for each outcome in the three-coin case, there are now 2^{n-3} outcomes, each corresponding to a possible configuration of the 4th flip and beyond. Since A, B, C, and D do not care about the outcomes of the 4th flip and beyond, this means that the size of each set is simply multiplied by 2^{n-3}. Therefore we have $|A| = 4 \times 2^{n-3} = 2^{n-1}$, $|B| = 4 \times 2^{n-3} = 2^{n-1}$, $|C| = 2 \times 2^{n-3} = 2^{n-2}$, and $|D| = 6 \times 2^{n-3} = 3 \times 2^{n-2}$.

(h) There are 6 possible outcomes which are all equally likely. We have 3 choices for the coin that we draw (which we represent by HH, HT and TT). Then for each coin we have two choices, we either see the first side or the second side (which we represent by 1 and 2). So the outcomes are $\Omega = \{(HH,1),(HH,2),(HT,1),(HT,2),(TT,1),(TT,2)\}$. Now given that we saw a heads, we can get rid of 3 of the outcomes, and the possible remaining outcomes are $\{(HH,1),(HH,2),(HT,1)\}$ which are all equally likely. In this space, the event that the coin has two heads is $\{(HH,1),(HH,2)\}$ which consists of two equally likely outcomes. So the probability is $2/3$.

2 Venn Diagram

Out of 1000 computer science students, 400 belong to a club (and may work part time), 500 work part time (and may belong to a club), and 50 belong to a club and work part time.

(a) Suppose we choose a student uniformly at random. Let C be the event that the student belongs to a club and P the event that the student works part time. Draw a picture of the sample space Ω and the events C and P.

(b) What is the probability that the student belongs to a club?

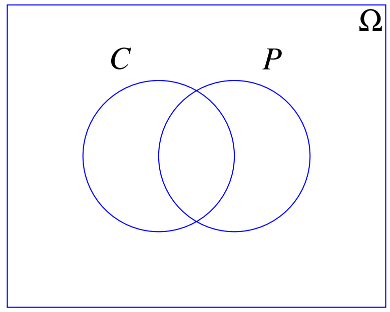
(c) What is the probability that the student works part time?

(d) What is the probability that the student belongs to a club AND works part time?

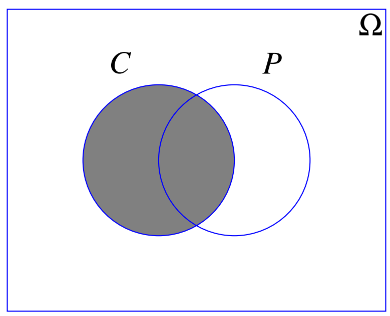
(e) What is the probability that the student belongs to a club OR works part time?

Solution:

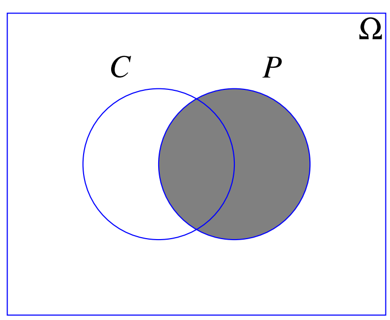
(a) The sample space will be illustrated by a Venn diagram.



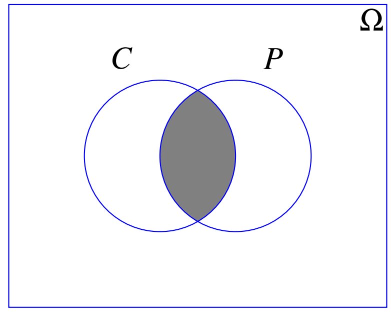
(b) $\mathbb{P}[C] = \frac{|C|}{|\Omega|} = \frac{400}{1000} = \frac{2}{5}$.



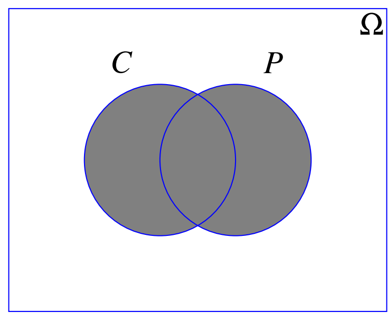
(c) $\mathbb{P}[P] = \frac{|P|}{|\Omega|} = \frac{500}{1000} = \frac{1}{2}$.



(d) $\mathbb{P}[P \cap C] = \frac{|P \cap C|}{|\Omega|} = \frac{50}{1000} = \frac{1}{20}$.



(e) $\mathbb{P}[P \cup C] = \mathbb{P}[P] + \mathbb{P}[C] - \mathbb{P}[P \cap C] = \frac{1}{2} + \frac{2}{5} - \frac{1}{20} = \frac{17}{20}$.



3 Counting & Probability

Consider the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 70$, where each x_i is a non-negative integer. We choose one of these solutions uniformly at random.

(a) What is the size of the sample space?

(b) What is the probability that both $x_1 \geq 30$ and $x_2 \geq 30$?

(c) What it the probability that either $x_1 \geq 30$ or $x_2 \geq 30$?

Solution:

(a) $\binom{75}{5}$. This is stars and bars.

(b) Put 30 balls each into the x_1 bin and the x_2 bin. We are left with 10 balls to distribute, whence there are $\binom{15}{5}$ possibilities. So the probability is $\binom{15}{5}/\binom{75}{5}$.

(c) Let A_i be the event that $x_i \geq 30$, then by inclusion-exclusion $\mathbb{P}[A_1 \cup A_2] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B] = \left[\binom{45}{5} + \binom{45}{5} - \binom{15}{5}\right] / \binom{75}{5}$.