1 Set Operations

- $\mathbb{R}$, the set of real numbers
- $\mathbb{Q}$, the set of rational numbers: $\{a/b : a, b \in \mathbb{Z} \wedge b \neq 0\}$
- $\mathbb{Z}$, the set of integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{N}$, the set of natural numbers: $\{0, 1, 2, 3, \ldots\}$

(a) Given a set $A = \{1,2,3,4\}$, what is $\mathscr{P}(A)$ (Power Set)?

(b) Given a generic set B, how do you describe $\mathscr{P}(B)$ using set comprehension notation? (Set Comprehension is $\{x \mid x \in A\}$.)

(c) What is $\mathbb{R} \cap \mathscr{P}(A)$?

(d) What is $\mathbb{R} \cap \mathbb{Z}$?

(e) What is $\mathbb{N} \cup \mathbb{Q}$?

(f) What kind of numbers are in $\mathbb{R} \setminus \mathbb{Q}$?

(g) If $S \subseteq T$, what is $S \setminus T$?

Solution:

(a)

$$\mathscr{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$

(b) $\mathscr{P}(B) = \{T \mid T \subseteq B\}$

(c) $\{\}$ or $\varnothing$

(d) $\mathbb{Z}$

(e) $\mathbb{Q}$

(f) The set of irrational numbers

(g) $\varnothing$

2 Image and Preimage

Let X and Y be sets, and $f : X \to Y$ be a function. For a subset, $A \subseteq X$, define it's image to be $f(A) = \{f(x) \mid x \in A\}$. For a subset $B \subseteq Y$, define it's preimage $f^{-1}(B) = \{x \mid f(x) \in B\}$. Note that in this context f^{-1} does not refer to an inverse function, as f may not have an inverse.

(a) Let $B \subseteq F(X)$. Prove that $f(f^{-1}(B)) = B$

(b) Let $A \subseteq X$. Prove that $A \subseteq f^{-1}(f(A))$

(c) Give an example of when $A \neq f^{-1}(f(A))$

(d) Suppose f is injective. Is it true that $A = f^{-1}(f(A))$? Prove or provide a counter-example.

Solution:

(a) First note that $f^{-1}(B) \subseteq X$. Suppose $x \in f^{-1}(B)$. Then by definition, $f(x) \in B$, so $f(f^{-1}(B)) \subseteq B$. Now suppose $y \in B$. Since $B \subseteq F(X)$ there exists an $x \in X$ such that $f(x) = y$, and such an x must be in $f^{-1}(B)$. Since $y = f(x)$, it must be that $y \in f(f^{-1}(B))$ so $B \subseteq f(f^{-1}(B))$. Thus, $B = f(f^{-1}(B))$.

(b) Suppose $x \in A$. By definition, $f(x) \in f(A)$, and so $x \in f^{-1}(f(A))$

(c) Let $X = \{-1, 1\}$ and $Y = \mathbb{R}$. Consider the function $f(x) = x^2$. Let $A = \{1\}$. Here, $f(A) = \{1\}$ and $f^{-1}(A) = \{-1, 1\} \neq A$.

(d) Suppose $x \in f^{-1}(f(A))$. Then there exists a $y \in f(A)$ such that $f(x) = y$. Since $y \in f(A)$, there exists $x' \in A$ such that $f(x') = y$. Since f is injective, $x' = x$, which means $x \in A$. Thus, $f^{-1}(f(A)) \subseteq A$. By part b, we have already shown the previous containment, so this means $f^{-1}(f(A)) = A$.

3 Bijections

Consider the function

$$f(x) = \begin{cases} x, & \text{if } x \geq 1; \\ x^2, & \text{if } -1 \leq x < 1; \\ 2x + 3, & \text{if } x < -1. \end{cases}$$

(a) If the domain and range of f are $\mathbb{N}$, is f injective (one-to-one), surjective (onto), bijective?

(b) If the domain and range of f are $\mathbb{Z}$, is f injective (one-to-one), surjective (onto), bijective?

(c) If the domain and range of f are $\mathbb{R}$, is f injective (one-to-one), surjective (onto), bijective?

Solution:

(a) Yes, Yes, Yes: On $\mathbb{N}$, f is simply the identity function $id(x) = x$.

(b) No, No, No: Both -1 and 1 get mapped to 1 (hence not injective) and there is no $x \in \mathbb{Z}$ that gets mapped to -2 (hence not surjective).

(c) No, Yes, No: -1 and 1 still get mapped to 1 (hence not injective), but every value can be attained (since f is a continuous function and $\lim_{x\to\pm\infty} f(x) = \pm\infty$), so f is surjective.