1 Prove or Disprove

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

Solution:

(a) **Answer**: True.
Proof: We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 4n$, we get $(2k + 1)^2 + 4 \times (2k + 1)$. Simplifying the expression yields $4k^2 + 12k + 5$. This can be rewritten as $2 \times (2k^2 + 6k + 2) + 1$. Since $2k^2 + 6k + 2$ is a natural number, by the definition of odd numbers, $n^2 + 4n$ is odd.
Alternatively, we could also factor the expression to get $n(n+4)$. Since n is odd, $n+4$ is also odd. The product of 2 odd numbers is also an odd number. Hence $n^2 + 4n$ is odd.

(b) **Answer**: True.
Proof: We will use a proof by contraposition. Suppose that $a > 11$ and $b > 4$ (note that this is equivalent to $\neg\big(a \leq 11 \vee b \leq 4\big)$). Since $a > 11$ and $b > 4$, $a + b > 15$ (note that $a + b > 15$ is equivalent to $\neg\big(a + b \leq 15\big)$). Thus, if $a + b \leq 15$, then $a \leq 11$ or $b \leq 4$.

(c) **Answer**: True.
Proof: We will use a proof by contraposition. Assume that r is rational. Since r is rational, it can be written in the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$. Then r^2 can be written as $\frac{a^2}{b^2}$. By the definition of rational numbers, r^2 is a rational number, since both a^2 and b^2 are integers, with $b \neq 0$. By contraposition, if r^2 is irrational, then r is irrational.

(d) **Answer**: False.
Proof: We will use proof by counterexample. Let $n = 7$. $5 \times 7^3 = 1715$. $7! = 5040$. Since $5n^3 < n!$, the claim is false.

2 Pigeonhole Principle

Prove the following statement: If you put $n+1$ balls into n bins, however you want, then at least one bin must contain at least two balls. This is known as the *pigeonhole principle*.

Solution:

We will use a proof by contradiction. Suppose this is not the case. Then all the bins would contain at most one ball. Then the maximum number of balls we could have would be n, but this is a contradiction since we have $n+1$ balls.

3 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \dots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.

Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0, 1, ..., n-1$. The objects assigned to these containers are the people at the party. However, containers 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.