1 Implication

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.

(b) $\exists x \exists y P(x,y) \implies \exists y \exists x P(x,y)$.

(c) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$.

(d) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.

Solution:

(a) True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.

(b) True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

(c) False. Let $P(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.

(d) True. The first statement says that there is an x, say x' where for every y, $P(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note: 4c and 4d are not logically equivalent. In fact, the converse of 4d is 4c, which we saw is false.

2 XOR

The truth table of XOR (denoted by $\oplus$) is as follows.

1. Express XOR using only $(\land, \lor, \neg)$ and parentheses.
2. Does $(A \oplus B)$ imply $(A \lor B)$? Explain briefly.
3. Does $(A \lor B)$ imply $(A \oplus B)$? Explain briefly.

A	B	A $\oplus$ B
F	F	F
F	T	T
T	F	T
T	T	F

Solution:

1. These are all correct:
   - $A \oplus B = (A \wedge \neg B) \vee (\neg A \wedge B)$
     Notice that there are only two instances when $A \oplus B$ is true: (1) when A is true and B is false, or (2) when B is true and A is false. The clause $(A \wedge \neg B)$ is only true when (1) is, and the clause $(\neg A \wedge B)$ is only true when (2) is.
   - $A \oplus B = (A \vee B) \wedge (\neg A \vee \neg B)$
     Another way to think about XOR is that exactly one of A and B needs to be true. This also means exactly one of $\neg A$ and $\neg B$ needs to be true. The clause $(A \vee B)$ tells us *at least* one of A and B needs to be true. In order to ensure that one of A or B is also false, we need the clause $(\neg A \vee \neg B)$ to be satisfied as well.
   - $A \oplus B = (A \vee B) \wedge \neg(A \wedge B)$
     This is the same as the previous, with De Morgan's law applied to equate $(\neg A \vee \neg B)$ to $\neg(A \wedge B)$.
2. Yes. $(A \oplus B) \implies (A \wedge \neg B) \vee (\neg A \wedge B) \implies (A \vee B)$. When $(A \oplus B)$ is true, at least one of A or B is true, which makes $(A \vee B)$ true as well.
3. No. When A and B are both true, then $(A \vee B)$ is true, but $(A \oplus B)$ is false.

3 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \equiv P \wedge Q$

(b) $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$

(c) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$

Solution:

(a) Not equivalent.

P	Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	F	F

(b) Equivalent.

P	Q	R	$(P \lor Q) \land R$	$(P \land R) \lor (Q \land R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(c) Equivalent.

P	Q	R	$(P \land Q) \lor R$	$(P \lor R) \land (Q \lor R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	F	F

4 Converse and Contrapositive

Consider the statement "if a natural number is divisible by 4, it is divisible by 2".

(a) Write the statement in propositional logic. Prove that it is true or give a counterexample.

(b) Write the inverse of the implication in English and in propositional logic. Prove that it is true or give a counterexample. (The inverse of an implication $P \implies Q$ is $\neg P \implies \neg Q$.)

(c) Write the converse of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

(d) Write the contrapositive of the implication in English and in propositional logic. Prove that it is true or give a counterexample.

Solution:

(a) $(\forall x \in \mathbb{N})\ (4 \mid x \implies 2 \mid x)$. This statement is true. We know that if x is divisible by 4, we can write x as $4k$ for some integer k. But $4k = (2 \cdot 2)k = 2(2k)$, where $2k$ is also an integer. Thus, x must also be divisible by 2, since it can be written as 2 times an integer.

(b) The inverse is that if a natural number is not divisible by 4, it is not divisible by 2: $(\forall x \in \mathbb{N})\ (4 \nmid x \implies 2 \nmid x)$. This is false, since 2 is not divisible by 4, but is divisible by 2.

(c) The converse is that any natural number that is divisible by 2 is also divisible by 4: $(\forall x \in \mathbb{N})\ (2 \mid x \implies 4 \mid x)$. Again, this is false, since 2 is divisible by 2 but not by 4.

(d) The contrapositive is that any natural number that is not divisible by 2 is not divisible by 4: $(\forall x \in \mathbb{N})\ (2 \nmid x \implies 4 \nmid x)$. To show that this is true, first consider that saying that x is not divisible by 2 is equivalent to saying that $x/2$ is not an integer. And if we divide a non-integer by an integer, we get back another non-integer–so $(x/2)/2 = x/4$ must also not be an integer. But that is exactly the same as saying that x is not divisible by 4.

Note that the inverse and the converse will always be contrapositives of each other, and so will always be logically equivalent.