CS 70 Discrete Mathematics and Probability Theory
Spring 2021 Discussion 13B

1 Interesting Gaussians

(a) If $X \sim N(0,\sigma_X^2)$ and $Y \sim N(0,\sigma_Y^2)$ are independent, then what is $\mathbb{E}\left[(X+Y)^k\right]$ for any *odd* $k \in \mathbb{N}$?

(b) Let $f_{\mu,\sigma}(x)$ be the density of a $N(\mu,\sigma^2)$ random variable, and let X be distributed according to $\alpha f_{\mu_1,\sigma_1}(x)+(1-\alpha)f_{\mu_2,\sigma_2}(x)$ for some $\alpha \in [0,1]$. Please compute $\mathbb{E}[X]$ and $\mathrm{Var}[X]$. Is X normally distributed?

Solution:

(a) $\boxed{\mathbb{E}\left[(X+Y)^k\right]=0}$.

Since X and Y are Gaussians, so must $Z=X+Y$ be. Moreover, as Z is of mean 0, we know that its distribution f_Z is symmetric around the origin, i.e. $f_Z(x)=f_Z(-x)$ for any $a,b \in \mathbb{R}$. Therefore, $\mathbb{E}\left[(X+Y)^k\right]=\mathbb{E}\left[Z^k\right]=\int_{-\infty}^{\infty} x^k f_z(x)\ \mathrm{d}x=\int_{-\infty}^{0} x^k f_Z(x)\ \mathrm{d}x+\int_0^\infty x^k f_Z(x)\ \mathrm{d}x=\int_0^\infty (-x)^k f_Z(-x)\ \mathrm{d}x+\int_0^\infty x^k f_Z(x)\ \mathrm{d}x=-\int_0^\infty x^k f_Z(x)\ \mathrm{d}x+\int_0^\infty x^k f_Z(x)\ \mathrm{d}x=0$, since k is odd.

(b) $\boxed{\mathbb{E}[X]=\alpha\mu_1+(1-\alpha)\mu_2}$

$\boxed{\mathrm{Var}[X]=\alpha\left(\sigma_1^2+\mu_1^2\right)+(1-\alpha)\left(\sigma_2^2+\mu_2^2\right)-(\mathbb{E}[X])^2}$

$\boxed{\text{No, } X \text{ is not necessarily normally distributed}}$.

$$\begin{aligned}
\mathbb{E}[X] := \mu &= \int_{-\infty}^{\infty} x\left(\alpha f_{\mu_1,\sigma_1}(x)+(1-\alpha)f_{\mu_2,\sigma_2}(x)\right)\ \mathrm{d}x \\
&= \alpha\int_{-\infty}^{\infty} x f_{\mu_1,\sigma_1}(x)\ \mathrm{d}x+(1-\alpha)\int_{-\infty}^{\infty} x f_{\mu_2,\sigma_2}(x)\ \mathrm{d}x=\alpha\mu_1+(1-\alpha)\mu_2 \\
\mathrm{Var}[X] := \sigma^2 = \mathbb{E}\left[X^2\right]-\mu^2 &= \alpha\int_{-\infty}^{\infty} x^2 f_{\mu_1,\sigma_1}(x)\ \mathrm{d}x+(1-\alpha)\int_{-\infty}^{\infty} x^2 f_{\mu_2,\sigma_2}(x)\ \mathrm{d}x-\mu^2 \\
&= \alpha\left(\sigma_1^2+\mu_1^2\right)+(1-\alpha)\left(\sigma_2^2+\mu_2^2\right)-\mu^2.
\end{aligned}$$

We know that the density of $N(\mu,\sigma)$ has a unique maximum at $x=\mu$; however, if, e.g. $\alpha=1/2, \mu_1=-10, \mu_2=10, \sigma_1=\sigma_2=1$, then $\alpha f_{\mu_1,\sigma_1}+(1-\alpha)f_{\mu_2,\sigma_2}$ has two maxima, and so cannot be the density of a Gaussian.

2 Erasures, Bounds, and Probabilities

Alice is sending 1000 bits to Bob. The probability that a bit gets erased is p, and the erasure of each bit is independent of the others.

Alice is using a scheme that can tolerate up to one-fifth of the bits being erased. That is, as long as Bob receives at least 801 of the 1000 bits correctly, he can decode Alice's message.

In other words, Bob becomes unable to decode Alice's message only if 200 or more bits are erased. We call this a "communication breakdown", and we want the probability of a communication breakdown to be at most 10^{-6}.

(a) Use Chebyshev's inequality to upper bound p such that the probability of a communications breakdown is at most 10^{-6}.

(b) As the CLT would suggest, approximate the fraction of erasures by a Gaussian random variable (with suitable mean and variance). Use this to find an approximate bound for p such that the probability of a communications breakdown is at most 10^{-6}.

You may use that $\Phi^{-1}(1-10^{-6}) \approx 4.753$.

Solution:

(a) Chebyshev's inequality states the following:

$$\Pr(\|X-\mu_X\| \geq k\sigma_X) \leq \frac{1}{k^2}$$

So we need to choose a k given by:

$$k = \frac{200-1000p}{\sqrt{1000p(1-p)}}$$

Note: The above is valid only for $200-1000p > 0$, or $p < 0.2$, since k has to be positive. But as we will see below, our upper bound for p will be below 0.2, so there is no problem.

Proceeding with the above value of k, and substituting for μ_X and σ_X, we obtain:

$$\Pr(\|X-1000p\| \geq 200-1000p) \leq \frac{1}{\left(\frac{(200-1000p)^2}{1000p(1-p)}\right)}$$

Simplifying, we get:

$$\Pr(\|X-1000p\| \geq 200-1000p) \leq \frac{p(1-p)}{40(1-5p)^2}$$

Now we know the following:

$$\begin{aligned}\Pr(X \geq 200) &= \Pr(X - 1000p \geq 200 - 1000p)\\ &\leq \Pr(\|X - 1000p\| \geq 200 - 1000p)\\ &\leq \frac{p(1-p)}{40(1-5p)^2}\end{aligned}$$

As before, to meet our objective, we just have to ensure that

$$\frac{p(1-p)}{40(1-5p)^2} \leq 10^{-6},$$

which yields an upper bound of about 3.998×10^{-5} for p.

(b) Let Y be equal to the fraction of erasures, i.e. $\frac{X}{1000}$. Using properties of expectation and variance, we can see that

$$\begin{aligned}\mathbb{E}[Y] &= p\\ \operatorname{Var}(Y) &= \operatorname{Var}(X) \cdot \frac{1}{1000^2} = \frac{p(1-p)}{1000}\end{aligned}$$

Therefore, by Central Limit Theorem, we can say that Y is roughly a normal distribution with that mean and variance. Since we are interested in the event that $Y \geq 0.2$, let's figure out how many standard deviations above the mean 0.2 is:

$$\frac{0.2-p}{\sqrt{\frac{p(1-p)}{1000}}} = \frac{(0.2-p)\sqrt{1000}}{\sqrt{p(1-p)}}.$$

Therefore, the probability that we get a failure should be approximately (by CLT),

$$1 - \Phi\left(\frac{(0.2-p)\sqrt{1000}}{\sqrt{p(1-p)}}\right)$$

where Φ is the CDF of a standard normal variable. Setting this to be at most 10^{-6} gives us

$$\Phi\left(\frac{(0.2-p)\sqrt{1000}}{\sqrt{p(1-p)}}\right) \geq 1 - 10^{-6}$$

And, since $\Phi^{-1}(1-10^{-6}) \approx 4.753$, we solve the inequality

$$\frac{(0.2-p)\sqrt{1000}}{\sqrt{p(1-p)}} \geq 4.753$$

This yields that we need $p \leq 0.1468$.

Note that this gives quite a different value from the previous parts. This is because the Central Limit Theorem gives a much tighter approximation for tail events than Markov's and Chebyshev's. Therefore, we do not need p to be so low to achieve a communication breakdown probability of 10^{-6}. The other bounds required us to need a probability of on the order of 10^{-5}, but here we realize that we only need it to be less than 0.1468. Quite drastic!

3 Suspicious Envelopes

There are two sealed envelopes. One containing x dollars and the other one containing $2x$ dollars. You select one of the two envelopes at random (but, don't open it).

(a) According to the logic below, you should keep swapping the selected envelope with the other one indefinitely to improve your expected earning. Is something wrong in this logic?

Logic: Let F and S denote the the amount in the envelope you select at random and the other one, respectively.Then, $S = 2F$ or $\frac{F}{2}$, with equal probability $\frac{1}{2}$. Hence, $\mathbb{E}[S] = \frac{1}{2}\left(2F + \frac{F}{2}\right) = 1.25F$, and you are better off exchanging your selected (and still sealed) envelope with the the other sealed envelope. The same logic is applicable after the exchange, and swapping should continue ad infinitum.

(b) You are now allowed to pick one envelope and see how much cash is inside, and then based on this information, you can decide to switch envelopes or stick with the envelope you already have.

Can you come up with a strategy which will allow you to pick the envelope with more money, with probability strictly greater than $1/2$?

Solution:

(a) The statement "$\mathbb{E}[S] = \frac{1}{2}\left(2F + \frac{F}{2}\right) = 1.25F$" in the logic on the previous page can't be correct. $\mathbb{E}[S]$ is a constant parameter, and F is a non-constant random variable. The correct logic should go as follow: $\mathbb{E}[S|F = x] = 2x$ and $\mathbb{E}[S|F = 2x] = x$. By smoothing property of conditional expectation, $\mathbb{E}[S] = \frac{1}{2}E[S|F = x] + \frac{1}{2}E[S|F = 2x] = \frac{1}{2}(2x + x) = 1.5x$. Also, $\mathbb{E}[F] = 1.5x$. Hence, from the standpoint of the expected value, there's no advantage in exchanging the second envelope for the first one.

(b) Surprisingly, the answer is yes.

First, we are going to choose a random positive integer z. To accomplish this, we flip coins! Let n be the number of flips you need before you see heads. Then, set $z = 2^{n-1} + 0.5$ (okay we lied; this is not an integer, but the 0.5 is just there to break ties later on).

Here is the strategy: look at how much cash is in the envelope you pick. If the amount of cash in the envelope exceeds z, then keep the envelope; otherwise, switch to the other envelope.

Suppose z is smaller than the amount of money in either envelope. Then, you will always keep your original envelope, but since the envelope you chose in the first place is equally likely to be the more lucrative envelope, you still have a $1/2$ probability of choosing the right envelope.

A similar analysis holds if z is greater than the amount of money in either envlope; here you will always switch envelopes, and you have a $1/2$ probability of choosing the right envelope.

However, what happens if z happens to land in between the values of the envelopes? If you initially chose the envelope with less money, then you will switch to the better envelope; if you initially chose the envlope with more money, you will stick with the better envelope. So, in this case, you are guaranteed to end up with the better envelope!

Since z has a positive probability of landing between the values of the envelopes, this strategy gives you a probability of choosing the better envelope which is strictly better than $1/2$.