CS 70 Discrete Mathematics and Probability Theory
Spring 2021 Discussion 13A

1 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \le x \le 1, 0 \le y \le 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal distribution of X.

(c) Find the conditional distribution of Y given $X = x$.

(d) Are X and Y independent?

Solution:

(a) Since $f(x,y)$ is a probability density function, it must integrate to 1. Then:

$$1 = \int_0^1 \int_0^2 Cxy\,dy\,dx = \int_0^1 2Cxdx = C$$

Therefore, $C = 1$.

(b) To get the marginal distribution of X, we integrate the joint distribution with respect to Y. So:

$$f_X(x) = \int_0^2 f(x,y)dy = \int_0^2 xydy = 2x$$

This is the marginal distribution for $0 \le x \le 1$.

(c) The conditional distribution of Y given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xy}{2x} = \frac{y}{2}$$

(d) The conditional distribution of Y given $X = x$ does not depend on x, so they are independent.

Alternatively, you could find the marginal distribution of Y and see it is the same as the conditional distribution of Y:

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 xydx = \frac{y}{2}$$

Notice that since X and Y are independent, $f_X(x)f_Y(y) = xy = f_{X,Y}(x,y)$, i.e. the product of the marginal distributions is the same as the joint distribution.

2 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

Solution:

First we write down what we want and expand out the conditioning:

$$\mathbb{P}[X \geq 5 \mid Y \geq X] = \frac{\mathbb{P}[Y \geq X \cap X \geq 5]}{\mathbb{P}[Y \geq X]}.$$

$\mathbb{P}[Y \geq X] = 1/2$ by symmetry. To find $\mathbb{P}[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\mathbb{P}[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

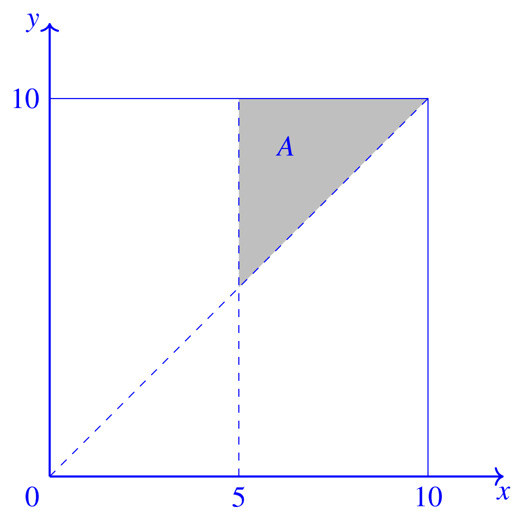

Figure 1: Joint probability density for the spinner.

$$\mathbb{P}[Y \geq X \cap X \geq 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8}.$$

So $\mathbb{P}[X \geq 5 \mid Y \geq X] = (1/8)/(1/2) = 1/4$.

3 Exponential Practice

(a) Let $X_1, X_2 \sim \text{Exponential}(\lambda)$ be independent, $\lambda > 0$. Calculate the density of $Y := X_1 + X_2$. [*Hint*: One way to approach this problem would be to compute the CDF of Y and then differentiate the CDF.]

(b) Let $t > 0$. What is the density of X_1, conditioned on $X_1 + X_2 = t$? [*Hint*: Once again, it may be helpful to consider the CDF $\mathbb{P}(X_1 \le x \mid X_1 + X_2 = t)$. To tackle the conditioning part, try conditioning instead on the event $\{X_1 + X_2 \in [t, t+\varepsilon]\}$, where $\varepsilon > 0$ is small.]

Solution:

(a) Let $y > 0$. Observe that if $X_1 + X_2 \le y$, then since $X_1, X_2 \ge 0$, it follows that $X_1 \le y$ and $X_2 \le y - X_1$.

$$\begin{aligned}\mathbb{P}(Y \le y) = \mathbb{P}(X_1 \le y, X_2 \le y - X_1) &= \int_0^y \int_0^{y-x_1} \lambda \exp(-\lambda x_1) \lambda \exp(-\lambda x_2) \, \mathrm{d}x_2 \, \mathrm{d}x_1 \\ &= \lambda^2 \int_0^y \exp(-\lambda x_1) \cdot \frac{1 - \exp(-\lambda(y - x_1))}{\lambda} \, \mathrm{d}x_1 \\ &= \lambda \int_0^y \big(\exp(-\lambda x_1) - \exp(-\lambda y)\big) \, \mathrm{d}x_1 = \lambda \Big(\frac{1 - \exp(-\lambda y)}{\lambda} - y \exp(-\lambda y)\Big).\end{aligned}$$

Upon differentiating the CDF, we have

$$\begin{aligned}f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} \mathbb{P}(Y \le y) &= \lambda \exp(-\lambda y) - \lambda \exp(-\lambda y) + \lambda^2 y \exp(-\lambda y) \\ &= \lambda^2 y \exp(-\lambda y), \qquad y > 0.\end{aligned}$$

Alternative solution: Since X_1 and X_2 are limits of X_1^n/n and X_2^n/n, where X_1^n and X_2^n are independent Geom$(p_n = \lambda/n)$, we know that $f_Y(y)\mathrm{d}y = \lim_{n\to\infty} \mathbb{P}\left[(X_1^n + X_2^n)/n = y\right]$, i.e. $f(y) = \lim_{n\to\infty} n\mathbb{P}\left[X_1^n + X_2^n = ny\right]$. But from worksheet 11b we know that

$$n\mathbb{P}\left[X_1^n + X_2^n = ny\right] = n(ny-1)(1-p_n)^{ny-2} p_n^2 = \lambda^2 \left(y - \frac{1}{n}\right)\left(1 - \frac{\lambda}{n}\right)^{ny-2},$$

which as $n \to \infty$ converges to $\lambda^2 y e^{-\lambda y}$ as desired.

(b) Let $0 \le x \le t$. Following the hint, we have

$$\begin{aligned}\mathbb{P}(X_1 \le x \mid X_1 + X_2 \in [t, t+\varepsilon]) &= \frac{\mathbb{P}(X_1 \le x, X_1 + X_2 \in [t, t+\varepsilon])}{\mathbb{P}(X_1 + X_2 \in [t, t+\varepsilon])} \\ &= \frac{\mathbb{P}(X_1 \le x, X_2 \in [t - X_1, t - X_1 + \varepsilon])}{f_Y(t) \cdot \varepsilon} \\ &= \frac{\int_0^x \int_{t-x_1}^{t-x_1+\varepsilon} \lambda \exp(-\lambda x_1) \lambda \exp(-\lambda x_2) \, \mathrm{d}x_2 \, \mathrm{d}x_1}{\lambda^2 t \exp(-\lambda t) \cdot \varepsilon} \\ &= \frac{\lambda^2 \int_0^x \exp(-\lambda x_1) \exp(-\lambda(t - x_1)) \varepsilon \, \mathrm{d}x_1}{\lambda^2 t \exp(-\lambda t) \cdot \varepsilon} = \frac{\int_0^x \mathrm{d}x_1}{t} = \frac{x}{t}.\end{aligned}$$

This means that the density is

$$f_{X_1 \mid X_1 + X_2}(x \mid t) = \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{P}(X \le x \mid X_1 + X_2 = t) = \frac{1}{t}, \qquad x \in [0, t],$$

which means that conditioned on $X_1 + X_2 = t$, X_1 is actually uniform on the interval $[0,t]$!

Alternative solution: Using the discrete approximations X_1^n/n and X_2^n/n as in the alternative solution to part (a), we have

$$
\begin{aligned}
n \cdot \mathbb{P}(X_1^n = xn \mid X_1^n + X_2^n = tn) &= n \frac{\mathbb{P}(X_1^n = xn \cap X_2^n = tn - xn)}{\mathbb{P}(X_1^n + X_2^n = tn)} = n \frac{(1-p_n)^{xn-1} p_n (1-p_n)^{tn-xn-1} p_n}{(tn-1)(1-p_n)^{tn-2} p_n^2} \\
&= \frac{1}{t - 1/n},
\end{aligned}
$$

which converges to $1/t$ as $n \to \infty$ just like before.