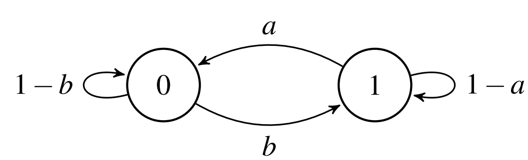
CS 70 Discrete Mathematics and Probability Theory
Spring 2021 **Discussion 12B**

1 Markov Chain Terminology

In this question, we will walk you through terms related to Markov chains.

1. (Irreducibility) A Markov chain is irreducible if, starting from any state i, the chain can transition to any other state j, possibly in multiple steps.
2. (Periodicity) $d(i) := \gcd\{n > 0 \mid P^n(i,i) = \mathbb{P}[X_n = i \mid X_0 = i] > 0\}$, $i \in \mathscr{X}$. If $d(i) = 1\ \forall i \in \mathscr{X}$, then the Markov chain is aperiodic; otherwise it is periodic.
3. (Matrix Representation) Define the transition probability matrix P by filling entry (i, j) with probability $P(i, j)$.
4. (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equations: $\pi = \pi P$.



(a) For what values of a and b is the above Markov chain irreducible? Reducible?

(b) For $a = 1$, $b = 1$, prove that the above Markov chain is periodic.

(c) For $0 < a < 1$, $0 < b < 1$, prove that the above Markov chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

(e) Write down the balance equations for this Markov chain and solve them. Assume that the Markov chain is irreducible.

Solution:

Full solutions will be released alongside Homework 12 solutions.

2 Allen's Umbrella Setup

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix.

(b) What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

Solution:

(a) Suppose Allen is in state 0. Then, Allen has no umbrellas to bring, so with probability 1 Allen arrives at a location with 2 umbrellas. That is,

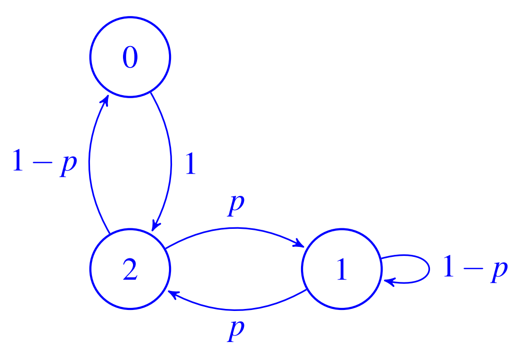
$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 0] = 1.$$

Suppose Allen is in state 1. With probability p, it rains and Allen brings the umbrella, arriving at state 2. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 1.

$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 1] = p, \qquad \mathbb{P}[X_{n+1} = 1 \mid X_n = 1] = 1-p$$

Suppose Allen is in state 2. With probability p, it rains and Allen brings the umbrella, arriving at state 1. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 0.

$$\mathbb{P}[X_{n+1} = 1 \mid X_n = 2] = p, \qquad \mathbb{P}[X_{n+1} = 0 \mid X_n = 2] = 1-p$$



We summarize this with the transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

(b) The transition matrices would be expressed as P^2 and P^n. Below we find the stationary distribution.

Observe that the transition matrix has non-zero element in its diagonal, which means the minimum number of steps to transit to state 1 from itself is one. Thus this transition matrix is irreducible and aperiodic, so it converges to its invariant distribution. To solve for the distribution, we set $\pi P = \pi$, or $\pi(P-I) = 0$. This yields the balance equations

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ 0 & -p & p\\ 1-p & p & -1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}.$$

As usual, one of the equations is redundant. We replace the last column by the normalization condition $\pi(0)+\pi(1)+\pi(2) = 1$.

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ 0 & -p & 1\\ 1-p & p & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

Now solve for the distribution:

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix} = \frac{1}{3-p}\begin{bmatrix}1-p & 1 & 1\end{bmatrix}$$

The invariant distribution also tells us the long-term fraction of time that Allen spends in each state. We can see that Allen spends a fraction $(1-p)/(3-p)$ of his time with no umbrella in his location, so the long-term fraction of time in which he walks through rain is $p(1-p)/(3-p)$.

3 Consecutive Flips

Suppose you are flipping a fair coin (one Head and one Tail) until you get the same side 3 times (Heads, Heads, Heads) or (Tails, Tails, Tails) in a row.

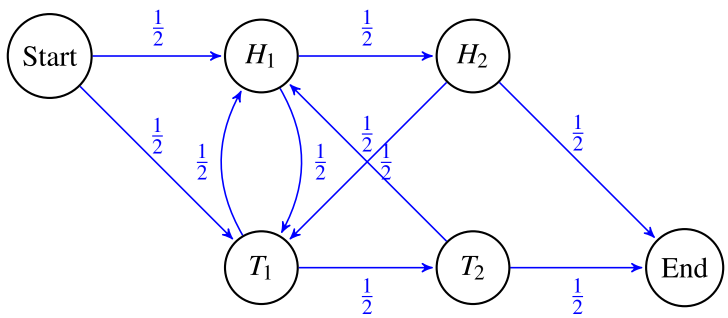
(a) Construct an Markov chain that describes the situation with a start state and end state.

(b) Given that you have flipped a (Tails, Heads) so far, what is the expected number of flips to see the same side three times?

(c) What is the expected number of flips to see the same side three times, beginning at the start state?

Solution:

(a) The appropriate Markov chain has 6 states: Start, H_1, H_2, T_1, T_2, and End.
For starting node, there is an outgoing edge to H_1 and T_1, each with equal probability of $1/2$.
For H_1, there is an outgoing edge to H_2 and T_1, each with equal probability of $1/2$.
For H_2, there is an outgoing edge to End and T_1, each with equal probability of $1/2$.
For T_1, there is an outgoing edge to H_1 and T_2, each with equal probability of $1/2$.
For T_2, there is an outgoing edge to H_1 and End, each with equal probability of $1/2$.



(b) If you got a Tails and then a Heads, you are currently in the H_1 state. Thus, we must calculate the expected number of flips to end from H_1. Thus we will do this with a system of equations. Since we are not trying to solve for the starting state, we have 5 unknowns that depend on 5 linearly independent equations. Let $\beta(i)$ be the expected number of flips to reach the end state starting from state i. Then we have:

$$\begin{aligned}
\beta(H_1) &= 1 + 0.5\beta(T_1) + 0.5\beta(H_2) \\
\beta(H_2) &= 1 + 0.5\beta(\text{End}) + 0.5\beta(T_1) \\
\beta(T_1) &= 1 + 0.5\beta(T_2) + 0.5\beta(H_1) \\
\beta(T_2) &= 1 + 0.5\beta(\text{End}) + 0.5\beta(H_1) \\
\beta(\text{End}) &= 0
\end{aligned}$$

If we solve this system of equations, we get $\beta(H_1) = 6, \beta(H_2) = 4, \beta(T_1) = 6, \beta(T_2) = 4$.

(c) $\beta(S) = 1 + 0.5\beta(H_1) + 0.5\beta(T_1) = 1 + 0.5 \cdot 6 + 0.5 \cdot 6 = 7$.