# 1 Number Game

Sinho and Vrettos are playing a game where they each choose an integer uniformly at random from $[0,100]$, then whoever has the larger number wins (in the event of a tie, they replay). However, Vrettos doesn't like losing, so he's rigged his random number generator such that it instead picks randomly from the integers between Sinho's number and 100. Let $S$ be Sinho's number and $V$ be Vrettos' number.

(a) What is $\mathbb{E}[S]$?

(b) What is $\mathbb{E}[V|S=s]$, where $s$ is any constant such that $0 \leq s \leq 100$?

(c) What is $\mathbb{E}[V]$?

**Solution:**

(a) $S$ is a (discrete) uniform random variable between 0 and 100, so its expectation is $\frac{0+100}{2} = 50$.

(b) If $S = s$, we know that $V$ will be uniformly distributed between $s$ and 100. Similar to the previous part, this gives us that $\mathbb{E}[V|S=s] = \frac{s+100}{2}$.

(c) Using the previous part and the Law of Total Expectation, we get

$$\begin{aligned}\mathbb{E}[V] = \mathbb{E}[\mathbb{E}[V \mid S]] = \mathbb{E}\left[\frac{S+100}{2}\right] \\ = \frac{\mathbb{E}[S]+100}{2} \\ = \frac{150}{2} = 75.\end{aligned}$$

**Alternate Solution:**
We have that

$$\begin{aligned}\mathbb{E}[V] &= \sum_{s=0}^{100} \mathbb{E}[V|S=s] \cdot \mathbb{P}[S=s] \\ &= \sum_{s=0}^{100} \frac{s+100}{2} \cdot \frac{1}{101} \\ &= \frac{1}{202}\left(\sum_{s=0}^{100} s + \sum_{s=0}^{100} 100\right)\end{aligned}$$

The first summation comes out to $\frac{100(100+1)}{2} = 50 \cdot 101$; the second summation is just adding 100 to itself 101 times, so it comes out to $100 \cdot 101$. Plugging these values in, we get $\mathbb{E}[V] = 75$.

## 2 Number of Ones

In this problem, we will revisit dice-rolling, except with conditional expectation.

(a) If we roll a die until we see a 6, how many ones should we expect to see? Notice that this is the MMSE of the number of ones we see given that we've seen a 6.

(b) If we roll a die until we see a number greater than 3, how many ones should we expect to see?

**Solution:**

(a) Let $Y$ be the number of ones we see. Let $X$ be the number of rolls we take until we get a 6.

Let us first compute $\mathbb{E}[Y \mid X]$. We know that in each of our $k-1$ rolls before the $k$th, we necessarily roll a number in $\{1, 2, 3, 4, 5\}$. Thus, we have a $1/5$ chance of getting a one, meaning

$$\mathbb{E}[Y \mid X = k] = \frac{1}{5}(k-1)$$

so

$$\mathbb{E}[Y \mid X] = \frac{1}{5}(X-1).$$

If this is confusing, write $Y$ as a sum of indicator variables.

$$Y = Y_1 + Y_2 + \cdots + Y_k$$

where $Y_i$ is 1 if we see a one on the $i$th roll. This means

$$\mathbb{E}[Y \mid X = k] = \mathbb{E}[Y_1 \mid X = k] + \mathbb{E}[Y_2 \mid X = k] + \cdots + \mathbb{E}[Y_k \mid X = k].$$

We know for a fact that on the $k$th roll, we roll a 6, thus $\mathbb{E}[Y_k] = 0$. Thus, we actually consider

$$\begin{aligned}\mathbb{E}[Y_1 \mid X = k] + \mathbb{E}[Y_2 \mid X = k] + \cdots + \mathbb{E}[Y_{k-1} \mid X = k] &= (k-1)\,\mathbb{E}[Y_1 \mid X = k] \\ &= (k-1)\mathbb{P}[Y_1 = 1 \mid X = k] \\ &= (k-1)\frac{1}{5}.\end{aligned}$$

Using the Law of Total Expectation, we know that

$$\begin{aligned}\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]] &= \mathbb{E}\left[\frac{1}{5}(X-1)\right] \\ &= \frac{1}{5}\,\mathbb{E}[X-1] \\ &= \frac{1}{5}(\mathbb{E}[X]-1).\end{aligned}$$

Since, $X \sim \text{Geometric}(1/6)$, the expected number of rolls until we roll a 6 is $\mathbb{E}[X] = 6$.

$$\frac{1}{5}(\mathbb{E}[X] - 1) = \frac{1}{5}(6-1) = 1.$$

(b) We use the same logic as the first part, except now each of the first $k-1$ rolls can only be 1, 2, or 3, so

$$\mathbb{E}[Y \mid X = k] = \frac{1}{3}(k-1).$$

Then

$$\begin{aligned}\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]] &= \mathbb{E}\left[\frac{1}{3}(X-1)\right] \\ &= \frac{1}{3}(\mathbb{E}[X] - 1).\end{aligned}$$

Since $X \sim \text{Geometric}(1/2)$, we know that the expected number of rolls until we roll a number greater than 3 is $\mathbb{E}[X] = 2$. This makes $\mathbb{E}[Y] = 1/3$.

# 3 Marbles in a Bag

We have $r$ red marbles, $b$ blue marbles, and $g$ green marbles in the same bag. If we sample marbles with replacement until we get 3 red marbles (not necessarily consecutively), how many blue marbles should we expect to see? (*Hint*: It might be useful to use Law of Total Expectation, $E(Y) = E(E(Y|X))$ )

**Solution:**

Let $Y$ be the number of blue marbles we see. Let $X$ be the samples we take until we get 3 red marbles.

Let us first compute $\mathbb{E}[Y \mid X]$. Let $Y_i$ be 1 if we see a blue marble on the $i$th sample and $Y = \sum_{i=1}^{k} Y_i$. This means

$$\begin{aligned}\mathbb{E}[Y \mid X = k] &= \mathbb{E}\left[\sum_{i=1}^{k} Y_i \,\middle|\, X = k\right] \\ &= \sum_{i=1}^{k} \mathbb{E}[Y_i \mid X = k].\end{aligned}$$

However, three $Y_i$ (call them $Y_a, Y_b, Y_c$) have $\mathbb{E}[Y_i] = 0$, since there are necessarily 3 red marbles. This means the other $k-3$ marbles are necessarily blue or green.

$$\begin{aligned}\sum_{i \neq a,b,c} \mathbb{E}[Y_i \mid X = k] &= \sum_{i \neq a,b,c} \mathbb{P}[Y_i = 1 \mid X = k] \\ &= \sum_{i \neq a,b,c} \frac{b}{b+g} \\ &= (k-3)\frac{b}{b+g}.\end{aligned}$$

This means

$$\mathbb{E}[Y \mid X] = (X-3)\frac{b}{b+g}.$$

Using the Law of Total Expectation, we know that

$$\begin{aligned}\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y \mid X]] &= \mathbb{E}\left[\frac{b}{b+g}(X-3)\right] \\ &= \frac{b}{b+g}\mathbb{E}[X-3] \\ &= \frac{b}{b+g}(\mathbb{E}[X]-3).\end{aligned}$$

We notice that

$$X = X_1 + X_2 + X_3,$$

where each $X_i$ represents the number of marbles seen between drawing the $(i-1)$th and $i$th red marble. We know that the absolute number of marbles seen between 2 consecutive red marbles is geometric, since we want to find the number of draws until the first red marble.

$$X_i \sim \text{Geometric}\left(\frac{r}{r+b+g}\right)$$

Since $X_1, X_2, X_3$ are identically distributed, we know that

$$\mathbb{E}[X] = 3\,\mathbb{E}[X_i] = 3(r+g+b)/r$$

.

$$\mathbb{E}[Y] = \frac{b}{b+g}\left(3\frac{r+g+b}{r} - 3\right) = \frac{3b}{r}.$$

**Alternate Solution**:

We know that the absolute number of marbles $N$ seen between 2 consecutive red marbles is geometric, since we want to find the number of draws until the first red marble. And given the number of marbles, $N$, between 2 consecutive red marbles, the number of blue marbles among these is distributed binomially.

Therefore, each $X_i$ is drawn from a binomial distribution, where the number of trials is distributed geometrically.

We exclude the very last marble in the binomial distribution, because we know it must be red (and therefore cannot be blue). And the probability for the binomial is $b/(b+g)$ because we know that in between 2 consecutive red balls, we can only have blue or green balls. So,

$$X_i \sim \text{Binomial}\left(\frac{b}{b+g}, N-1\right), \qquad \text{where} \qquad N \sim \text{Geometric}\left(\frac{r}{r+b+g}\right).$$

And, applying the law of conditional expectation, we have

$$\begin{aligned}
\mathbb{E}(X_i) &= \mathbb{E}(\mathbb{E}(X_i \mid N)) \\
&= \mathbb{E}\left((N-1)\frac{b}{b+g}\right) \\
&= \frac{b}{b+g}\,\mathbb{E}(N-1) \\
&= \frac{b}{b+g}\left(\frac{r+b+g}{r}-1\right) \\
&= \frac{b}{b+g}\left(\frac{b+g}{r}\right) \\
&= \frac{b}{r}.
\end{aligned}$$

We know that each of the $X_i$'s is identically distributed, so

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 3 \cdot \mathbb{E}(X_1) = \frac{3b}{r}.$$