CS 70 Discrete Mathematics and Probability Theory
Fall 2020 HW 1

1 Calculus Review

(a) Compute a closed-form expression for the value of following summation:

$$\sum_{k=1}^{\infty} \frac{9}{2^k}$$

(b) Use summation notion to write an expression equivalent to the following statement:

The sum of the first n consecutive odd integers, starting from 1

(c) Compute the following integral:

$$\int_0^{\infty} \sin(t)e^{-t}dt$$

(d) Find the maximum value of the following function and determine where it occurs:

$$f(x) = -x \cdot \ln x$$

Solution:

(a) Use the convergence of geometric series with $|r| < 1$.

$$\sum_{k=1}^{\infty} \frac{9}{2^k} = 9 \cdot \sum_{k=1}^{\infty} \frac{1}{2^k} = 9 \cdot (\sum_{k=0}^{\infty} \frac{1}{2^k} - 1)$$
$$= 9 \cdot (2-1) = 9$$

(b) Observe that $2k+1$ is odd for all $k \in \mathbb{Z}$.

$$\sum_{k=0}^{n-1} 2k+1$$

(c) Let $I = \int sin(t)e^{-t}$.
Use integration by parts, with $u = \sin(t)$ and $dv = e^{-t}$.
This means $du = \cos(t)$ and $v = -e^{-t}$.

$$I = \int \sin(t)e^{-t}dt = uv - \int v \cdot du$$
$$= -\sin(t)e^{-t} + \int e^{-t}\cos(t)dt$$

Use integration by parts again on $\int e^{-t}\cos(t)dt$, with $u=\cos(t)$ and $dv=e^{-t}$. This means $du=-\sin(t)$ and $v=-e^{-t}$.

$$\begin{aligned}\int e^{-t}\cos(t)dt &= uv-\int v\cdot du\\ &= -\cos(t)e^{-t}-\int e^{-t}\cdot\sin(t)dt\\ &= -\cos(t)e^{-t}-I\end{aligned}$$

Combining these results:

$$\begin{aligned}I &= -\sin(t)e^{-t}-\cos(t)e^{-t}-I\\ \Rightarrow 2I &= -\sin(t)e^{-t}-\cos(t)e^{-t}\\ \Rightarrow I &= \frac{-\sin(t)e^{-t}-\cos(t)e^{-t}}{2}\end{aligned}$$

Finally, we have:

$$I\Big|_0^\infty = \frac{0-0}{2}-\frac{0-1}{2}=\frac{1}{2}$$

(d) Compute the derivative of the function, and set it equal to 0.

$$\begin{aligned}\frac{df}{dx} &= -1\cdot\ln x+-x\cdot\frac{1}{x}\\ &= -\ln x-1=0\\ &\Rightarrow x^*=\frac{1}{e}\end{aligned}$$

The optimal value is achieved at $x^*=\frac{1}{e}$, and the corresponding value is $f(x^*)=\frac{1}{e}$.

2 Propositional Practice

In parts (a)-(c), convert the English sentences into propositional logic. In parts (d)-(f), convert the propositions into English. In part (f), let $P(a)$ represent the proposition that a is prime.

(a) There is one and only one real solution to the equation $x^2=0$.

(b) Between any two distinct rational numbers, there is another rational number.

(c) If the square of an integer is greater than 4, that integer is greater than 2 or it is less than -2.

(d) $(\forall x\in\mathbb{R})\ (x\in\mathbb{C})$

(e) $(\forall x,y\in\mathbb{Z})(x^2-y^2\neq 10)$

(f) $(\forall x\in\mathbb{N})\ [\ (x>1)\implies(\exists a,b\in\mathbb{N})\ ((a+b=2x)\wedge P(a)\wedge P(b))\ ]$

Solution:

(a) Let $p(x) = x^2$. The sentence can be read: "There is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x". Or,

$$(\exists x \in \mathbb{R})\left((p(x) = 0) \land ((\forall y \in \mathbb{R})(p(y) = 0) \implies (x = y))\right).$$

(b) The sentence can be read "If x and y are distinct rational numbers, then there is a rational number z between x and y." Or,

$$(\forall x, y \in \mathbb{Q})((x \neq y) \implies ((\exists z \in \mathbb{Q})(x < z < y \lor y < z < x))).$$

Equivalently,

$$(\forall x, y \in \mathbb{Q})((x = y) \lor (\exists z \in \mathbb{Q})(x < z < y \lor y < z < x)).$$

Note that $x < z < y$ is mathematical shorthand for $(x < z) \land (z < y)$, so the above statement is equivalent to

$$(\forall x, y \in \mathbb{Q})(x = y) \lor ((\exists z \in \mathbb{Q})\left((x < z) \land (z < y)\right) \lor ((y < z) \land (z < x))).$$

(c) $(\forall x \in \mathbb{Z})\ ((x^2 > 4) \implies ((x > 2) \lor (x < -2)))$

(d) All real numbers are complex numbers.

(e) There are no integer solutions to the equation $x^2 - y^2 = 10$.

(f) For any natural number greater than 1, there are some prime numbers a and b such that $2x = a + b$.

In other words: Any even integer larger than 2 can be written as the sum of two primes.

Aside: This statement is known as Goldbach's Conjecture, and it is a famous unsolved problem in number theory (https://xkcd.com/1310/).

3 Tautologies and Contradictions

Classify each statement as being one of the following, where P and Q are arbitrary propositions:

- True for all combinations of P and Q (Tautology)
- False for all combinations of P and Q (Contradiction)
- Neither

Justify your answers with a truth table.

(a) $P \implies (Q \land P) \lor (\neg Q \land P)$

(b) $(P \lor Q) \lor (P \lor \neg Q)$

(c) $P \land (P \implies \neg Q) \land (Q)$

(d) $(\neg P \implies Q) \implies (\neg Q \implies P)$

(e) $(\neg P \implies \neg Q) \land (P \implies \neg Q) \land (Q)$

(f) $(\neg(P \land Q)) \land (P \lor Q)$

Solution:

(a) **Tautology**

P	Q	$Q \land P$	$\neg Q \land P$	$P \implies (Q \land P) \lor (\neg Q \land P)$
T	T	T	F	T
T	F	F	T	T
F	T	F	F	T
F	F	F	F	T

(b) **Tautology**

P	Q	$P \lor Q$	$P \lor \neg Q$	$(P \lor Q) \lor (P \lor \neg Q)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(c) **Contradiction**

P	Q	$P \implies \neg Q$	$P \land (P \implies Q) \land (Q)$
T	T	F	F
T	F	T	F
F	T	T	F
F	F	T	F

(d) **Tautology**

P	Q	$\neg P \implies Q$	$\neg Q \implies P$	$(\neg P \implies Q) \implies (\neg Q \implies P)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

(e) **Contradiction**

P	Q	$P \implies \neg Q$	$\neg P \implies \neg Q$	$(P \implies \neg Q) \land (\neg P \implies \neg Q) \land (Q)$
T	T	F	T	F
T	F	T	T	F
F	T	T	F	F
F	F	T	T	F

(f) **Neither**

P	Q	$P \vee Q$	$\neg(P \wedge Q)$	$(P \vee Q) \wedge (\neg(P \wedge Q))$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

4 Prove or Disprove

For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

Solution:

(a) **Answer**: True.
Proof: We will use a direct proof. Assume n is odd. By the definition of odd numbers, $n = 2k + 1$ for some natural number k. Substituting into the expression $n^2 + 4n$, we get $(2k + 1)^2 + 4 \times (2k + 1)$. Simplifying the expression yields $4k^2 + 12k + 5$. This can be rewritten as $2 \times (2k^2 + 6k + 2) + 1$. Since $2k^2 + 6k + 2$ is a natural number, by the definition of odd numbers, $n^2 + 4n$ is odd.
Alternatively, we could also factor the expression to get $n(n+4)$. Since n is odd, $n+4$ is also odd. The product of 2 odd numbers is also an odd number. Hence $n^2 + 4n$ is odd.

(b) **Answer**: True.
Proof: We will use a proof by contraposition. Suppose that $a > 11$ and $b > 4$ (note that this is equivalent to $\neg\big(a \leq 11 \vee b \leq 4\big)$). Since $a > 11$ and $b > 4$, $a + b > 15$ (note that $a + b > 15$ is equivalent to $\neg\big(a + b \leq 15\big)$). Thus, if $a + b \leq 15$, then $a \leq 11$ or $b \leq 4$.

(c) **Answer**: True.
Proof: We will use a proof by contraposition. Assume that r is rational. Since r is rational, it can be written in the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$. Then r^2 can be written as $\frac{a^2}{b^2}$. By the definition of rational numbers, r^2 is a rational number, since both a^2 and b^2 are integers, with $b \neq 0$. By contraposition, if r^2 is irrational, then r is irrational.

(d) **Answer**: False.
Proof: We will use proof by counterexample. Let $n = 7$. $5 \times 7^3 = 1715$. $7! = 5040$. Since $5n^3 < n!$, the claim is false.

5 Twin Primes

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

Solution:

(a) First we note that any integer can be written in one of the forms $3k$, $3k+1$, or $3k+2$. (Note that $3k+2$ is equal to $3(k+1)-1$. Since k is arbitary, we can treat these as equivalent forms).

We can now prove the contrapositive: that any integer $m > 3$ of the form $3k$ must be composite. Any such integer is divisible by 3, so this is true right away. Thus our original claim is true as well.

(b) We can check all the primes up to 5 to see that of these, only 5 takes part in two twin prime pairs (3,5 and 5,7). What about primes > 5?

For any prime $m > 5$, we can check if $m+2$ and $m-2$ are both prime. Note that if $m > 5$, then $m+2 > 3$ and $m-2 > 3$ so we can apply part (a) and we can do a proof by cases based on the two forms from part (a).

Case 1: m is of the form $3k+1$. Then $m+2 = 3k+3$, which is divisible by 3. So $m+2$ is not prime.

Case 2: m is of the form $3k-1$. Then $m-2 = 3k-3$, which is divisible by 3. So $m-2$ is not prime.

So in either case, at least one of $m+2$ and $m-2$ is not prime.

6 Social Network

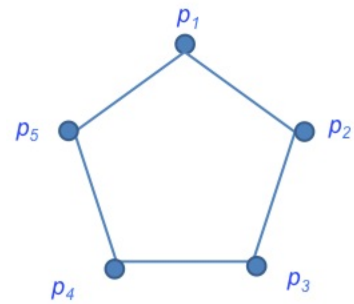
Consider the same setup as Q2 on the vitamin, where there are n people at a party, and every two people are either friends or strangers. Prove or provide a counterexample for the following statements.

(a) For all cases with $n = 5$ people, there exists a group of 3 people that are either all friends or all strangers.

(b) For all cases with $n = 6$ people, there exists a group of 3 people that are either all friends or all strangers.

Solution:

(a) The statement is false. A counterexample is shown below where people are connected if they are friends and unconnected if they are strangers. In this example, at most 2 are friends or strangers.



(b) The statement is true. We proceed with a proof by cases.

For any person p, we could divide the rest of people into 2 groups: the group of p's friends and the group of strangers. By pigeonhole principle, one of the groups must have at least 3 people.

Case 1a: p is friends with at least 3 people, and these friends are all strangers. Then p's friends form a group of at least 3 strangers.

Case 1b: p is friends with at least 3 people, and at least 2 of them are friends with each other. These two, along with p, form a group of 3 friends.

Case 2a: p is strangers with at least 3 people, and these strangers are all friends. Analogous to Case 1a, these strangers form a group of at least 3 friends.

Case 2b: p is strangers with at least 3 people, and at least 2 of them are not friends. Analogous to Case 1b, these 2 strangers form a group of at least strangers.

7 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Hint: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(c) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(d) $f(A \cup B) = f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(f) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

Solution:

In order to prove equality $A = B$, we need to prove that A is a subset of B, $A \subseteq B$ and that B is a subset of A, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose x is such that $f(x) \in A \cup B$. Then either $f(x) \in A$, in which case $x \in f^{-1}(A)$, or $f(x) \in B$, in which case $x \in f^{-1}(B)$, so in either case we have $x \in f^{-1}(A) \cup f^{-1}(B)$. This proves that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Suppose, without loss of generality, that $x \in f^{-1}(A)$. Then $f(x) \in A$, so $f(x) \in A \cup B$, so $x \in f^{-1}(A \cup B)$. The argument for $x \in f^{-1}(B)$ is the same. Hence, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

(b) Suppose x is such that $f(x) \in A \cap B$. Then $f(x)$ lies in both A and B, so x lies in both $f^{-1}(A)$ and $f^{-1}(B)$, so $x \in f^{-1}(A) \cap f^{-1}(B)$. So $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, x is in both $f^{-1}(A)$ and $f^{-1}(B)$, so $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. So $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

(c) Suppose x is such that $f(x) \in A \setminus B$. Then, $f(x) \in A$ and $f(x) \notin B$, which means that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which means that $x \in f^{-1}(A) \setminus f^{-1}(B)$. So $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, so $f(x) \in A$ and $f(x) \notin B$, so $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. So $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$.

(d) Suppose that $x \in A \cup B$. Then either $x \in A$, in which case $f(x) \in f(A)$, or $x \in B$, in which case $f(x) \in f(B)$. In either case, $f(x) \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now, suppose that $y \in f(A) \cup f(B)$. Then either $y \in f(A)$ or $y \in f(B)$. In the first case, there is an element $x \in A$ with $f(x) = y$; in the second case, there is an element $x \in B$ with $f(x) = y$. In either case, there is an element $x \in A \cup B$ with $f(x) = y$, which means that $y \in f(A \cup B)$. So $f(A) \cup f(B) \subseteq f(A \cup B)$.

(e) Suppose $x \in A \cap B$. Then, x lies in both A and B, so $f(x)$ lies in both $f(A)$ and $f(B)$, so $f(x) \in f(A) \cap f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Consider when there are elements $a \in A$ and $b \in B$ with $f(a) = f(b)$, but A and B are disjoint. Here, $f(a) = f(b) \in f(A) \cap f(B)$, but $f(A \cap B)$ is empty (since $A \cap B$ is empty).

(f) Suppose $y \in f(A) \setminus f(B)$. Since y is not in $f(B)$, there are no elements in B which map to y. Let x be any element of A that maps to y; by the previous sentence, x cannot lie in B. Hence, $x \in A \setminus B$, so $y \in f(A \setminus B)$. Hence, $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

Consider when $B = \{0\}$ and $A = \{0, 1\}$, with $f(0) = f(1) = 0$. One has $A \setminus B = \{1\}$, so $f(A \setminus B) = \{0\}$. However, $f(A) = f(B) = \{0\}$, so $f(A) \setminus f(B) = \varnothing$.