1 Markov Chains: Prove/Disprove

Prove or disprove the following statements, using the definitions from the previous question.

(a) There exists an irreducible, finite Markov chain for which there exist initial distributions that converge to different distributions.

(b) There exists an irreducible, aperiodic, finite Markov chain for which $\mathbb{P}(X_{n+1} = j \mid X_n = i) = 1$ or 0 for all i, j.

(c) There exists an irreducible, non-aperiodic Markov chain for which $\mathbb{P}(X_{n+1} = j \mid X_n = i) \neq 1$ for all i, j.

(d) For an irreducible, non-aperiodic Markov chain, any initial distribution not equal to the invariant distribution does not converge to any distribution.

Solution:

(a) False. Every finite irreducible Markov chain has a unique stationary distribution. If it's possible for the Markov chain to converge to two different distributions given different starting distributions, it implies there are two stationary distributions. To elaborate further, we know in the long run the fraction of time spent in each state converges to the stationary distribution. So if the distribution converges, the long-run fraction of time will be whatever distribution it converges to, which we see must be the stationary distribution.

(b) True, you can have one state pointing to itself. However for number of states > 1 it is false. Consider the initial distribution of having a probability of 1 of being in an arbitrary state. After a transition, the resulting distribution must be a probability 1 of being in a different state (if it were the same state, this would immediately imply that the Markov chain is reducible). Further transitions have the same effect. Therefore this initial distribution does not converge. Therefore this Markov chain cannot be aperiodic and irreducible (since it would converge in that case).

(c) True. Consider the states $\{0,1,2,3\}$. Set $P(i,j) = 1/2$ if $i \equiv j \pm 1 \mod 4$ and 0 otherwise. In other words, the Markov chain is a square with each side replaced with two links pointing in opposite directions with probabilities of $1/2$. Consider the period of state 0. Any path from 0 back to itself, such as $0-1-2-1-0$, alternates in parity of each consecutive state since each state only points to the state above or below it mod 4. Therefore state 0 has period 2. Therefore this Markov chain is not aperiodic (and all states have period 2).

(d) False. Take the initial distribution $\begin{bmatrix}0.25 & 0.30 & 0.25 & 0.20\end{bmatrix}$ for the above Markov chain. After one transition it goes to the invariant distribution, $\begin{bmatrix}0.25 & 0.25 & 0.25 & 0.25\end{bmatrix}$.

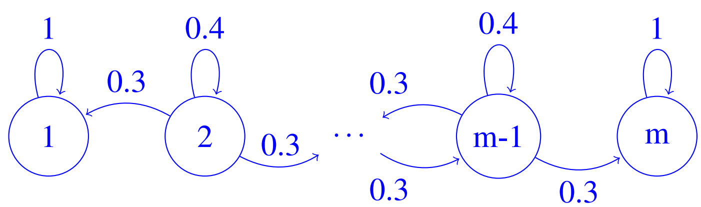
2 Can it be a Markov Chain?

(a) A fly flies in a straight line in unit-length increments. Each second it moves to the left with probability 0.3, right with probability 0.3, and stays put with probability 0.4. There are two spiders at positions 1 and m and if the fly lands in either of those positions it is captured. Given that hte fly starts between positions 1 and m, model this process as a Markov Chain.

(b) Take the same scenario as in the previous part with $m = 4$. Let $Y_n = 0$ if at time n the fly is in position 1 or 2 and let $Y_n = 1$ if at time n the fly is in position 3 or 4. Is the process Y_n a Markov chain?

Solution:

(a) We can draw the Markov chain as such:



(b) No, because the longer the fly stays in any one state, the more likely the fly gets in one of the absorbing states.

For example, say $\mathbb{P}[X_0 = 2] = \mathbb{P}[X_0 = 3] = 1/2$ and $\mathbb{P}[X_0 = 1] = \mathbb{P}[X_0 = 4] = 0$. Then

$$\begin{aligned}
\mathbb{P}[Y_2 = 0 \mid Y_1 = 1, Y_0 = 0] &= \mathbb{P}[X_2 \in \{1,2\} \mid X_1 = 3, X_0 = 2] \\
&= \mathbb{P}[X_2 = 2 \mid X_1 = 3] = 0.3 \\
\mathbb{P}[Y_2 = 0 \mid Y_1 = 1, Y_0 = 1] &= \mathbb{P}[Y_2 = 0, Y_1 = 1, Y_0 = 1] / \mathbb{P}[Y_1 = 1, Y_0 = 1] \\
&= \mathbb{P}[X_2 = 2, X_1 = 3, X_0 = 3] / (\mathbb{P}[X_1 = 3, X_0 = 3] + \mathbb{P}[X_1 = 4, X_0 = 3]) \\
&= \frac{0.5 \cdot 0.4 \cdot 0.3}{0.5 \cdot 0.4 + 0.5 \cdot 0.3} = \frac{6}{35}
\end{aligned}$$

If Y was Markov, then $\mathbb{P}[Y_2 = 0 \mid Y_1 = 1, Y_0 = 0] = \mathbb{P}[Y_2 = 0 \mid Y_1 = 1] = \mathbb{P}[Y_2 = 0 \mid Y_1 = 1, Y_0 = 1]$. However, $0.3 > 6/35$, and so Y cannot be Markov.

3 Allen's Umbrella Setup

Every morning, Allen walks from his home to Soda, and every evening, Allen walks from Soda to his home. Suppose that Allen has two umbrellas in his possession, but he sometimes leaves his umbrellas behind. Specifically, before leaving from his home or Soda, he checks the weather. If it is raining outside, he will bring his umbrella (that is, if there is an umbrella where he currently is). If it is not raining outside, he will forget to bring his umbrella. Assume that the probability of rain is p.

(a) Model this as a Markov chain. What is $\mathscr{X}$? Write down the transition matrix.

(b) What is the transition matrix after 2 trips? n trips? Determine if the distribution of X_n converges to the invariant distribution, and compute the invariant distribution. Determine the long-term fraction of time that Allen will walk through rain with no umbrella.

Solution:

(a) Suppose Allen is in state 0. Then, Allen has no umbrellas to bring, so with probability 1 Allen arrives at a location with 2 umbrellas. That is,

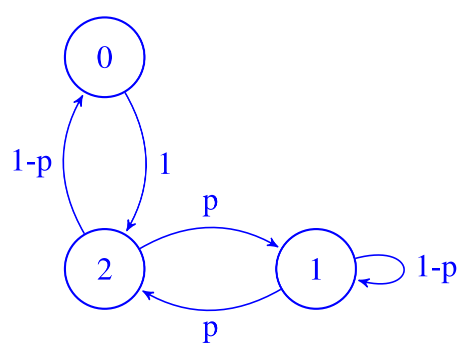
$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 0] = 1.$$

Suppose Allen is in state 1. With probability p, it rains and Allen brings the umbrella, arriving at state 2. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 1.

$$\mathbb{P}[X_{n+1} = 2 \mid X_n = 1] = p, \qquad \mathbb{P}[X_{n+1} = 1 \mid X_n = 1] = 1-p$$

Suppose Allen is in state 2. With probability p, it rains and Allen brings the umbrella, arriving at state 1. With probability $1-p$, Allen forgets the umbrella, so Allen arrives at state 0.

$$\mathbb{P}[X_{n+1} = 1 \mid X_n = 2] = p, \qquad \mathbb{P}[X_{n+1} = 0 \mid X_n = 2] = 1-p$$



We summarize this with the transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix}.$$

(b) The transition matrices would be expressed as P^2 and P^n. Below we find the stationary distribution.

Observe that the transition matrix has non-zero element in its diagonal, which means the minimum number of steps to transit to state 1 from itself is one. Thus this transition matrix is irreducible and aperiodic, so it converges to its invariant distribution. To solve for the distribution, we set $\pi P = \pi$, or $\pi(P - I) = 0$. This yields the balance equations

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ 0 & -p & p\\ 1-p & p & -1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}.$$

As usual, one of the equations is redundant. We replace the last column by the normalization condition $\pi(0) + \pi(1) + \pi(2) = 1$.

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix}\begin{bmatrix}-1 & 0 & 1\\ 0 & -p & 1\\ 1-p & p & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$$

Now solve for the distribution:

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix} = \frac{1}{3-p}\begin{bmatrix}1-p & 1 & 1\end{bmatrix}$$

The invariant distribution also tells us the long-term fraction of time that Allen spends in each state. We can see that Allen spends a fraction $(1-p)/(3-p)$ of his time with no umbrella in his location, so the long-term fraction of time in which he walks through rain is $p(1-p)/(3-p)$.

4 Three Tails

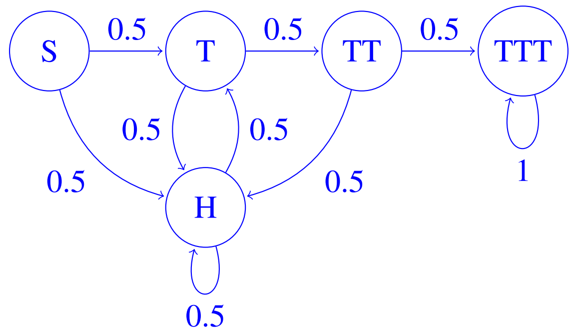
You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

Hint: How is this different than the number of *coins* flipped until getting TTT?

Solution:

We can model this problem as a Markov chain with the following states:

- S: Start state, which we are only in before flipping any coins.
- H: We see a head, which means no streak of tails currently exists.
- T: We've seen exactly one tail in a row so far.
- TT: We've seen exactly two tails in a row so far.
- TTT: We've accomplished our goal of seeing three tails in a row and stop flipping.



We can write the first step equations and solve for $\beta(S)$, only counting heads that we see since we are not looking for the total number of flips. The equations are as follows:

$$\begin{align}
\beta(S) &= 0.5\beta(T) + 0.5\beta(H) \tag{1}\\
\beta(H) &= 1 + 0.5\beta(H) + 0.5\beta(T) \tag{2}\\
\beta(T) &= 0.5\beta(TT) + 0.5\beta(H) \tag{3}\\
\beta(TT) &= 0.5\beta(H) + 0.5\beta(TTT) \tag{4}\\
\beta(TTT) &= 0 \tag{5}
\end{align}$$

From equation (2), we see that

$$0.5\beta(H) = 1 + 0.5\beta(T)$$

and can substitute that into equation (3) to get

$$0.5\beta(T) = 0.5\beta(TT) + 1.$$

Substituting this into equation (4), we can deduce that $\beta(TT) = 4$. This allows us to conclude that $\beta(T) = 6$, $\beta(H) = 8$, and $\beta(S) = 7$. On average, we expect to see 7 heads before flipping three tails in a row.