CS 70 Discrete Mathematics and Probability Theory
Fall 2020

DIS 13A

1 Continuous Joint Densities

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \le x \le 1, 0 \le y \le 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal distribution of X.

(c) Find the conditional distribution of Y given $X = x$.

(d) Are X and Y independent?

Solution:

(a) Since $f(x,y)$ is a probability density function, it must integrate to 1. Then:

$$1 = \int_0^1 \int_0^2 Cxy\,dy\,dx = \int_0^1 2Cx dx = C$$

Therefore, $C = 1$.

(b) To get the marginal distribution of X, we integrate the joint distribution with respect to Y. So:

$$f_X(x) = \int_0^2 f(x,y) dy = \int_0^2 xy dy = 2x$$

This is the marginal distribution for $0 \le x \le 1$.

(c) The conditional distribution of Y given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xy}{2x} = \frac{y}{2}$$

(d) The conditional distribution of Y given $X = x$ does not depend on x, so they are independent.

Alternatively, you could find the marginal distribution of Y and see it is the same as the conditional distribution of Y:

$$f_Y(y) = \int_0^1 f(x,y) dx = \int_0^1 xy dx = \frac{y}{2}$$

Notice that since X and Y are independent, $f_X(x) f_Y(y) = xy = f_{X,Y}(x,y)$, i.e. the product of the marginal distributions is the same as the joint distribution.

2 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

Solution:

First we write down what we want and expand out the conditioning:

$$\mathbb{P}[X \geq 5 \mid Y \geq X] = \frac{\mathbb{P}[Y \geq X \cap X \geq 5]}{\mathbb{P}[Y \geq X]}.$$

$\mathbb{P}[Y \geq X] = 1/2$ by symmetry. To find $\mathbb{P}[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\mathbb{P}[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

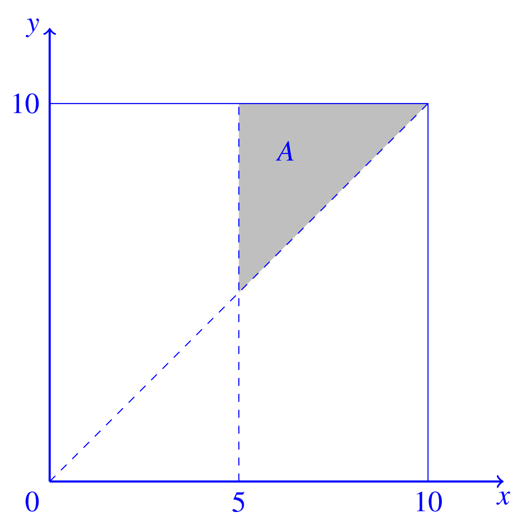

Figure 1: Joint probability density for the spinner.

$$\mathbb{P}[Y \geq X \cap X \geq 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8}.$$

So $\mathbb{P}[X \geq 5 \mid Y \geq X] = (1/8)/(1/2) = 1/4$.

3 Darts with Friends

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a circle of radius r around the center. Alex's aim follows a uniform distribution over a circle of radius $2r$ around the center.

(a) Let the distance of Michelle's throw be denoted by the random variable X and let the distance of Alex's throw be denoted by the random variable Y.

- What's the cumulative distribution function of X?
- What's the cumulative distribution function of Y?
- What's the probability density function of X?
- What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \min\{X,Y\}$?

(d) What's the cumulative distribution function of $V = \max\{X,Y\}$?

(e) What is the expectation of the absolute difference between Michelle's and Alex's distances from the center, that is, what is $\mathbb{E}[|X-Y|]$? [*Hint*: Use parts (c) and (d), together with the continuous version of the tail sum formula, which states that $\mathbb{E}[Z] = \int_0^\infty P(Z \geq z)dz$.]

Solution:

(a)
- To get the cumulative distribution function of X, we'll consider the ratio of the area where the distance to the center is less than x, compared to the entire available area. This gives us the following expression:

$$\mathbb{P}(X \leq x) = \frac{\pi x^2}{\pi r^2} = \frac{x^2}{r^2}, \qquad x \in [0,r]$$

- Using the same approach as the previous part:

$$\mathbb{P}(Y \leq y) = \frac{\pi y^2}{\pi \cdot 4r^2} = \frac{y^2}{4r^2}, \qquad y \in [0,2r]$$

- We'll take the derivative of the CDF to get the following:

$$f_X(x) = \frac{\mathrm{d}\mathbb{P}(X \leq x)}{\mathrm{d}x} = \frac{2x}{r^2}, \qquad x \in [0,r]$$

- Using the same approach as the previous part:

$$f_Y(y) = \frac{\mathrm{d}\mathbb{P}(Y \leq y)}{\mathrm{d}y} = \frac{y}{2r^2}, \qquad y \in [0,2r]$$

(b) We'll condition on Alex's outcome and then integrate over all the possibilities to get the marginal $\mathbb{P}(X \leq Y)$ as following:

$$\mathbb{P}(X \leq Y) = \int_0^{2r} \mathbb{P}(X \leq Y \mid Y = y) f_Y(y)\,\mathrm{d}y = \int_0^r \frac{y^2}{r^2} \times \frac{y}{2r^2}\,\mathrm{d}y + \int_r^{2r} 1 \times \frac{y}{2r^2}\,\mathrm{d}y$$
$$= \frac{r^4 - 0}{8r^4} + \frac{4r^2 - r^2}{4r^2} = \frac{1}{8} + \frac{3}{4} = \frac{7}{8}$$

Note the range within which $\mathbb{P}(X \le Y) = 1$. This allowed us to separate the integral to simplify our solution. Using this, we can get $\mathbb{P}(Y \le X)$ by the following:

$$\mathbb{P}(Y \le X) = 1 - \mathbb{P}(X \le Y) = \frac{1}{8}$$

A similar approach to the integral above could be used to verify this result.

$$\mathbb{P}(Y \le X) = \int_0^r \mathbb{P}(Y \le X \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^r \frac{x^2}{4r^2}\frac{2x}{r^2}\,\mathrm{d}x = \frac{1}{2r^4}\int_0^r x^3\,\mathrm{d}x = \frac{r^4}{8r^4} = \frac{1}{8}$$

(c) Getting the CDF of U relies on the insight that for the minimum of two random variables to be greater than a value, they both need to be greater than that value. Taking the complement of this will give us the CDF of U. This allows us to get the following result. For $u \in [0,r]$:

$$\begin{aligned}\mathbb{P}(U \le u) &= 1 - \mathbb{P}(U \ge u) = 1 - \mathbb{P}(X \ge u)\mathbb{P}(Y \ge u) = 1 - \big(1 - \mathbb{P}(X \le u)\big)\big(1 - \mathbb{P}(Y \le u)\big) \\ &= 1 - \left(1 - \frac{u^2}{r^2}\right)\left(1 - \frac{u^2}{4r^2}\right) = \frac{5u^2}{4r^2} - \frac{u^4}{4r^4}\end{aligned}$$

For $u > r$, we get $\mathbb{P}(X > u) = 0$, this makes $\mathbb{P}(U \le u) = 1$.

(d) Getting the CDF of V also relies on a similar insight that for the maximum of two random variables to be smaller than a value, they both need to be smaller than that value. Using this we can get the following result for $v \in [0,r]$:

$$\mathbb{P}(V \le v) = \mathbb{P}(X \le v)\mathbb{P}(Y \le v) = \left(\frac{v^2}{r^2}\right)\left(\frac{v^2}{4r^2}\right) = \frac{v^4}{4r^4}$$

For $v \in [r, 2r]$ we have $\mathbb{P}(X \le v) = 1$, this makes

$$\mathbb{P}(V \le v) = \mathbb{P}(Y \le v) = \frac{v^2}{4r^2}.$$

For $v > 2r$ we have $\mathbb{P}(V \le v) = 1$ since CDFs of both X and Y are 1 in this range.

(e) We can subtract U from V to get this difference. Using the tail-sum formula to calculate the expectation, we can get the following result:

$$\begin{aligned}\mathbb{E}[|X - Y|] = \mathbb{E}[V - U] = \mathbb{E}[V] - \mathbb{E}[U] &= \int_0^{2r} \mathbb{P}(V \ge v)\,\mathrm{d}v - \int_0^r \mathbb{P}(U \ge u)\,\mathrm{d}u \\ &= \int_0^r \left(1 - \frac{v^4}{4r^4}\right)\mathrm{d}v + \int_r^{2r}\left(1 - \frac{v^2}{4r^2}\right)\mathrm{d}v - \int_0^r \left(1 - \frac{5u^2}{4r^2} + \frac{u^4}{4r^4}\right)\mathrm{d}u \\ &= \frac{19r}{20} + \frac{5r}{12} - \frac{19r}{30} = \frac{11r}{15}\end{aligned}$$

Alternatively, you could derive the density of U and V and use those to calculate the expectation. For $v \in [0,r]$:

$$f_V(v) = \frac{\mathrm{d}\mathbb{P}(V \le v)}{\mathrm{d}v} = \frac{v^3}{r^4}$$

For $v \in [r, 2r]$:

$$f_V(v) = \frac{\mathrm{d}\mathbb{P}(V \le v)}{\mathrm{d}v} = \frac{v}{2r^2}$$

Using this we can calculate $\mathbb{E}[V]$ as:

$$\mathbb{E}[V] = \int_0^{2r} v f_V(v)\,\mathrm{d}v = \frac{1}{r^4}\int_0^r v^4\,\mathrm{d}v + \frac{1}{2r^2}\int_r^{2r} v^2\,\mathrm{d}v = \frac{r^5}{5r^4} + \frac{8r^3 - r^3}{6r^2} = \frac{r}{5} + \frac{7r}{6r} = \frac{41r}{30}$$

To calculate $\mathbb{E}[U]$ we will use the following PDF for $u \in [0, r]$:

$$f_U(u) = \frac{\mathrm{d}\mathbb{P}(U \le u)}{\mathrm{d}u} = \frac{5u}{2r^2} - \frac{u^3}{r^4}$$

We can get the $\mathbb{E}[U]$ by the following:

$$\mathbb{E}[U] = \int_0^r u f_U(u)\,\mathrm{d}u = \int_0^r \left(\frac{5u^2}{2r^2} - \frac{u^4}{r^4}\right)\mathrm{d}u = \frac{5r^3}{6r^2} - \frac{r^5}{5r^4} = \frac{5r}{6} - \frac{r}{5} = \frac{19r}{30}$$

Combining the two results gives us the same result as above:

$$\mathbb{E}[|X - Y|] = \mathbb{E}[V - U] = \mathbb{E}[V] - \mathbb{E}[U] = \frac{41r}{30} - \frac{19r}{30} = \frac{11r}{15}$$