CS 70 Discrete Mathematics and Probability Theory
Fall 2020 DIS 10B

1 Pullout Balls

Suppose you have a bag containing six balls numbered $1,2,3,4,5,6$.

(a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?

(b) You repeat the experiment from part (a), except this time you pull out two balls together and record the product of their numbers. What is the expected value of the total that you record?

Solution:

(a) Let X be the number that you record. Each ball is equally likely to be chosen, so

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}(X = x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5$$

As demonstrated here, the expected value of a random variable need not, and often is not, a feasible value of that random variable (there is no outcome ω for which $X(\omega) = 3.5$).

(b) Let Y be the product of two numbers that you pull out. Then

$$\mathbb{E}[Y] = \frac{\sum_{i=1}^{6} (i \times \sum_{j=i+1}^{6} j)}{\binom{6}{2}} = \frac{20+36+45+44+30}{15} = \frac{35}{3}$$

2 How Many Queens?

You shuffle a standard 52-card deck, before drawing the first three cards from the top of the pile. Let X denote the number of queens you draw.

(a) What is $\mathbb{P}(X=0)$, $\mathbb{P}(X=1)$, $\mathbb{P}(X=2)$ and $\mathbb{P}(X=3)$?

(b) What do your answers you computed in part a add up to?

(c) Compute $\mathbb{E}(X)$ from the definition of expectation.

(d) Are the X_i indicators independent?

Solution:

(a) Calculate each case of X = 0, 1, 2, 3:

We must draw 3 non-queen cards in a row, so the probability is

$$\mathbb{P}(X=0) = \frac{48}{52}\cdot\frac{47}{51}\cdot\frac{46}{50} = \frac{4324}{5525}.$$

Alternatively, every 3-card hand is equally likely, so we can use counting. There are $\binom{52}{3}$ total 3-card hands, and $\binom{48}{3}$ hands with only non-queen cards, which gives us the same result.

$$\mathbb{P}(X=0) = \frac{\binom{48}{3}}{\binom{52}{3}} = \frac{4324}{5525}$$

- We will continue to use counting. The number of hands with exactly one queen amounts to the number of ways to choose 1 queen out of 4, and 2 non-queens out of 48.

$$\mathbb{P}(X=1) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \frac{1128}{5525}$$

- Choose 2 queens out of 4, and 1 non-queen out of 48.

$$\mathbb{P}(X=2) = \frac{\binom{4}{2}\binom{48}{1}}{\binom{52}{3}} = \frac{72}{5525}$$

- Choose 3 queens out of 4.

$$\mathbb{P}(X=3) = \frac{\binom{4}{3}}{\binom{52}{3}} = \frac{1}{5525}$$

(b) We check:

$$\mathbb{P}(X=0)+\mathbb{P}(X=1)+\mathbb{P}(X=2)+\mathbb{P}(X=3) = \frac{4324+1128+72+1}{5525} = 1$$

(c) From the definition, $\mathbb{E}(X) = \sum_{k=0}^{3} k\mathbb{P}(X=k)$, so

$$\mathbb{E}(X) = 0\cdot\frac{4324}{5525} + 1\cdot\frac{1128}{5525} + 2\cdot\frac{72}{5525} + 3\cdot\frac{1}{5525} = \frac{3}{13}.$$

(d) No, they are not independent. As an example:

$$\mathbb{P}(X_1=1)\mathbb{P}(X_2=1) = \frac{1}{13}\cdot\frac{1}{13} = \frac{1}{169}$$

However,

$$\mathbb{P}(X_1=1, X_2=1) = \mathbb{P}(\text{the first and second cards are both queens}) = \frac{4}{52}\cdot\frac{3}{51} = \frac{1}{221}.$$

3 Head Count

Consider a coin with $\mathbb{P}(\text{Heads}) = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) Name the distribution of X and what its parameters are.

(b) What is $\mathbb{P}(X = 7)$?

(c) What is $\mathbb{P}(X \geq 1)$? Hint: You should be able to do this without a summation.

(d) What is $\mathbb{P}(12 \leq X \leq 14)$?

Solution:

(a) Since we have 20 independent trials, with each trial having a probability $2/5$ of success, $X \sim \text{Binomial}(20, 2/5)$.

(b)

$$\mathbb{P}(X = 7) = \binom{20}{7}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right)^{13}.$$

(c)

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d)

$$\begin{aligned}\mathbb{P}(12 \leq X \leq 14) &= \mathbb{P}(X = 12) + \mathbb{P}(X = 13) + \mathbb{P}(X = 14)\\ &= \binom{20}{12}\left(\frac{2}{5}\right)^{12}\left(\frac{3}{5}\right)^{8} + \binom{20}{13}\left(\frac{2}{5}\right)^{13}\left(\frac{3}{5}\right)^{7} + \binom{20}{14}\left(\frac{2}{5}\right)^{14}\left(\frac{3}{5}\right)^{6}.\end{aligned}$$