CS 70 Discrete Mathematics and Probability Theory
Fall 2020 DIS 00

1 Implication

Which of the following implications are always true, regardless of P? Give a counterexample for each false assertion (i.e. come up with a statement $P(x,y)$ that would make the implication false).

(a) $\forall x \forall y P(x,y) \implies \forall y \forall x P(x,y)$.

(b) $\forall x \exists y P(x,y) \implies \exists y \forall x P(x,y)$.

(c) $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$.

Solution:

(a) True. For all can be switched if they are adjacent; since $\forall x, \forall y$ and $\forall y, \forall x$ means for all x and y in our universe.

(b) False. Let $P(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $P(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.

(c) True. The first statement says that there is an x, say x' where for every y, $P(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note: 4c and 4d are not logically equivalent. In fact, the converse of 4d is 4c, which we saw is false.

2 Equivalences with Quantifiers

Evaluate whether the expressions on the left and right sides are equivalent in each part, and briefly justify your answers.

(a)	$\forall x \left((\exists y\, Q(x,y)) \Rightarrow P(x)\right)$	$\forall x\, \exists y \left(Q(x,y) \Rightarrow P(x)\right)$
(b)	$\neg \exists x\, \forall y \left(P(x,y) \Rightarrow \neg Q(x,y)\right)$	$\forall x \left((\exists y\, P(x,y)) \wedge (\exists y\, Q(x,y))\right)$
(c)	$\forall x\, \exists y \left(P(x) \Rightarrow Q(x,y)\right)$	$\forall x \left(P(x) \Rightarrow (\exists y\, Q(x,y))\right)$

Solution:

(a) Not equivalent.

Justification: We can rewrite the left side as as $\forall x \left((\neg(\exists y\, Q(x,y))) \vee P(x)\right)$ and the right side as $\forall x\, \exists y \left(\neg Q(x,y) \vee P(x)\right)$ Applying the negation on the left side of the equivalence

$(\neg(\exists y Q(x,y)))$ changes the $\exists y$ to $\forall y$, and the two sides are clearly not the same. Another approach to the problem is to consider by linguistic example. Let x and y span the universe of all people, and let $Q(x,y)$ mean "Person x is Person y's offspring", and let $P(x)$ mean "Person x likes tofu". The right side claims that, for all Persons x, there exists some Person y such that either Person x is not Person y's offspring or that Person x likes tofu. The left side claims that, for all Persons x, if there exists a parent of Person x, then Person x likes tofu. Obviously, these are not the same.

(b) Not equivalent.

Justification: Using De Morgan's Law to distribute the negation on the left side yields

$$\forall x\ \exists y\ \big(P(x,y) \wedge Q(x,y)\big).$$

But $\exists$ does not distribute over $\wedge$. There could exist different values of y such that $P(x,y)$ and $Q(x,y)$ for a given x, but not necessarily the same value.

(c) Equivalent.

Justification: We can rewrite the left side as $\forall x\ \exists y\ \big(\neg P(x) \vee Q(x,y)\big)$ and the right side as $\forall x\ \big(\neg P(x) \vee (\exists y\ Q(x,y))\big)$. Clearly, the two sides are the same if $\neg P(x)$ is true. If $\neg P(x)$ is false, then the two sides are still the same, because $\forall x\ \exists y\ \big(\text{False} \vee Q(x,y)\big) \equiv \forall x\ \big(\text{False} \vee (\exists y\ Q(x,y))\big)$.

3 XOR

The truth table of XOR (denoted by $\oplus$) is as follows.

A	B	A $\oplus$ B
F	F	F
F	T	T
T	F	T
T	T	F

1. Express XOR using only $(\wedge, \vee, \neg)$ and parentheses.
2. Does $(A \oplus B)$ imply $(A \vee B)$? Explain briefly.
3. Does $(A \vee B)$ imply $(A \oplus B)$? Explain briefly.

Solution:

1. These are all correct:
   - $A \oplus B = (A \wedge \neg B) \vee (\neg A \wedge B)$
     Notice that there are only two instances when $A \oplus B$ is true: (1) when A is true and B is false, or (2) when B is true and A is false. The clause $(A \wedge \neg B)$ is only true when (1) is, and the clause $(\neg A \wedge B)$ is only true when (2) is.

- $A \oplus B = (A \vee B) \wedge (\neg A \vee \neg B)$
  Another way to think about XOR is that exactly one of A and B needs to be true. This also means exactly one of $\neg A$ and $\neg B$ needs to be true. The clause $(A \vee B)$ tells us *at least* one of A and B needs to be true. In order to ensure that one of A or B is also false, we need the clause $(\neg A \vee \neg B)$ to be satisfied as well.
- $A \oplus B = (A \vee B) \wedge \neg(A \wedge B)$
  This is the same as the previous, with De Morgan's law applied to equate $(\neg A \vee \neg B)$ to $\neg(A \wedge B)$.

2. Yes. $(A \oplus B) \implies (A \wedge \neg B) \vee (\neg A \wedge B) \implies (A \vee B)$. When $(A \oplus B)$ is true, at least one of A or B is true, which makes $(A \vee B)$ true as well.

3. No. When A and B are both true, then $(A \vee B)$ is true, but $(A \oplus B)$ is false.

4 Truth Tables

Determine whether the following equivalences hold, by writing out truth tables. Clearly state whether or not each pair is equivalent.

(a) $P \wedge (Q \vee P) \equiv P \wedge Q$

(b) $(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)$

(c) $(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R)$

Solution:

(a) Not equivalent.

P	Q	$P \wedge (Q \vee P)$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	F	F
F	F	F	F

(b) Equivalent.

P	Q	R	$(P \vee Q) \wedge R$	$(P \wedge R) \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(c) Equivalent.

P	Q	R	$(P \wedge Q) \vee R$	$(P \vee R) \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	T	T
F	F	F	F	F