CS61B Lectures #28

Today:

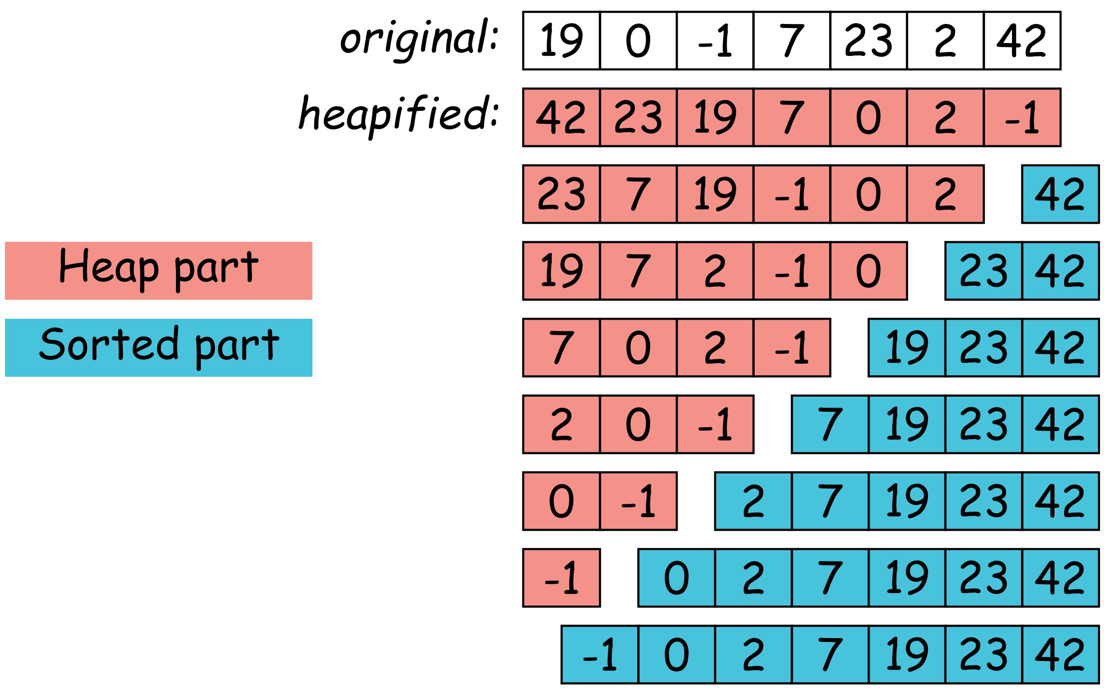
- Selection sorts, heap sort
- Merge sorts
- Quicksort

Readings: Today: *DS(IJ)*, Chapter 8; Next topic: Chapter 9.

Sorting by Selection: Heapsort

Idea: Keep selecting smallest (or largest) element.

- Really bad idea on a simple list or vector.
- But we've already seen it in action: use heap.
- Gives $O(N \lg N)$ algorithm (N remove-first operations).
- Since we remove items from end of heap, we can use that area to accumulate result:



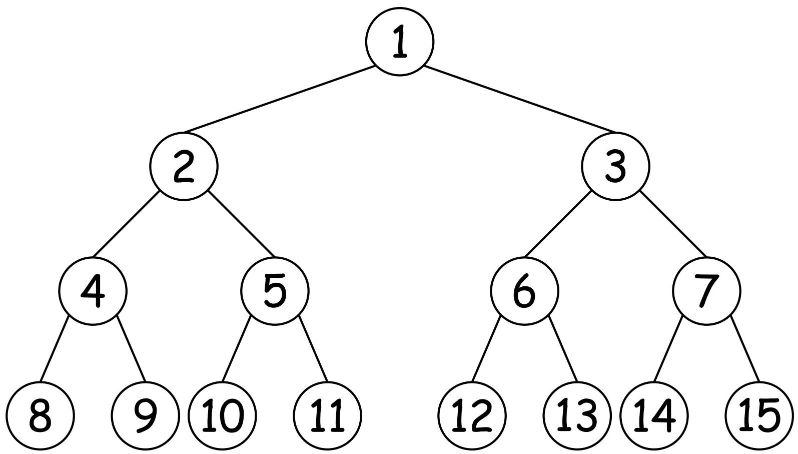
Sorting By Selection: Initial Heapifying

- When covering heaps before, we created them by insertion in an initially empty heap.
- When given an array of unheaped data to start with, there is a faster procedure (assume heap indexed from 0): [corrected 4/3]

```
void heapify(int[] arr) {
    int N = arr.length;
    for (int k = N / 2; k >= 0; k -= 1) {
        for (int p = k, c = 0; 2*p + 1 < N; p = c) {
            reheapify downward from p;
        }
    }
}
```

- At each iteration of the p loop, only the element at p might be out of order with respect to its descendants, so reheapifying downward will restore the subtree rooted at p to proper heap ordering.
- Looks like the procedure for re-inserting an element after the top element of the heap is removed, repeated $N/2$ times.
- But instead of being $\Theta(N \lg N)$, it's just $\Theta(N)$.

Cost of Creating Heap



1 node $\times$ 3 steps down

2 nodes $\times$ 2 steps down

4 nodes $\times$ 1 step down

- In general, worst-case cost for a heap with $h+1$ levels is

$$\begin{aligned}
& 2^0 \cdot h + 2^1 \cdot (h-1) + \ldots + 2^{h-1} \cdot 1 \\
= \; & (2^0 + 2^1 + \ldots + 2^{h-1}) + (2^0 + 2^1 + \ldots + 2^{h-2}) + \ldots + (2^0) \\
= \; & (2^h - 1) + (2^{h-1} - 1) + \ldots + (2^1 - 1) \\
= \; & 2^{h+1} - 1 - h \\
\in \; & \Theta(2^h) = \Theta(N)
\end{aligned}$$

- Alas, since the rest of heapsort still takes $\Theta(N \lg N)$, this does not improve its asymptotic cost.

Merge Sorting

Idea: Divide data in 2 equal parts; recursively sort halves; merge results.

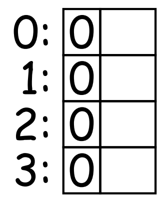
- Already seen analysis: $\Theta(N \lg N)$.
- Good for *external sorting:*
  - First break data into small enough chunks to fit in memory and sort.
  - Then repeatedly merge into bigger and bigger sequences.
- Can merge K sequences of *arbitrary size* on secondary storage using $\Theta(K)$ storage:

```
Data[] V = new Data[K];
For all i, set V[i] to the first data item of sequence i;
while there is data left to sort:
    Find k so that V[k] is smallest;
    Output V[k], and read new value into V[k] (if present).
```

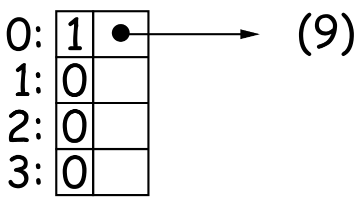
Illustration of Internal Merge Sort

For internal sorting, can use a *binomial comb* to orchestrate:

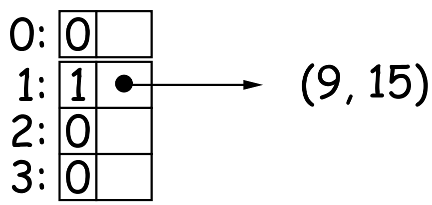
L: (9, 15, 5, 3, 0, 6, 10, -1, 2, 20, 8)



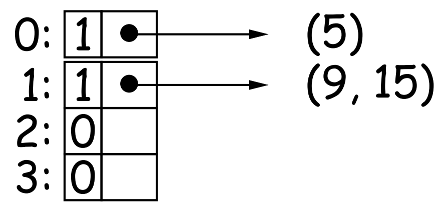
0 elements processed



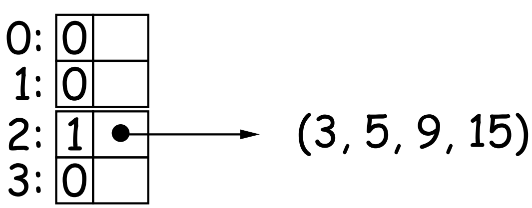
1 element processed



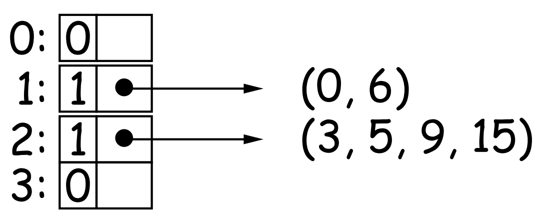
2 elements processed



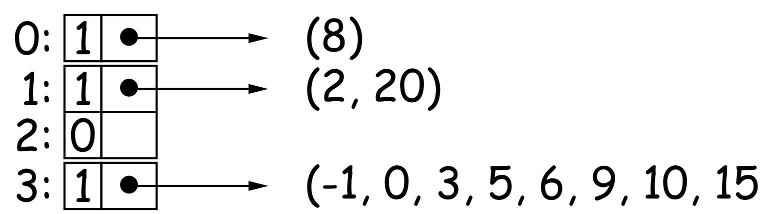
3 elements processed



4 elements processed



6 elements processed



11 elements processed

Quicksort: Speed through Probability

Idea:

- *Partition* data into pieces: everything $>$ a *pivot* value at the high end of the sequence to be sorted, and everything $\leq$ on the low end.
- Repeat recursively on the high and low pieces.
- For speed, stop when pieces are "small enough" and do insertion sort on the whole thing.
- Reason: insertion sort has low constant factors. By design, no item will move out of its piece [why?], so when pieces are small, #inversions is, too.
- Have to choose pivot well. E.g.: *median* of first, last and middle items of sequence.

Example of Quicksort

- In this example, we continue until pieces are size ≤ 4.
- Pivots for next step are starred. Arrange to move pivot to dividing line each time.
- Last step is insertion sort.

16	10	13	18	-4	-7	12	-5	19	15	0	22	29	34	-1*

-4	-5	-7	-1	18	13	12	10	19	15	0	22	29	34	16*

-4	-5	-7	-1	15	13	12*	10	0	16	19*	22	29	34	18

-4	-5	-7	-1	10	0	12	15	13	16	18	19	29	34	22

- Now everything is "close to" right, so just do insertion sort:

-7	-5	-4	-1	0	10	12	13	15	16	18	19	22	29	34

Performance of Quicksort

- Probabalistic time:
  - If choice of pivots good, divide data in two each time: $\Theta(N \lg N)$ with a good constant factor relative to merge or heap sort.
  - If choice of pivots bad, most items on one side each time: $\Theta(N^2)$.
  - $\Omega(N \lg N)$ in best case, so insertion sort better for nearly ordered input sets.
- Interesting point: randomly shuffling the data before sorting makes $\Omega(N^2)$ time *very* unlikely!

Quick Selection

The Selection Problem: for given k, find k^{th} smallest element in data.

- Obvious method: sort, select element #k, time $\Theta(N \lg N)$.
- If $k \leq$ some constant, can easily do in $\Theta(N)$ time:
  - Go through array, keep smallest k items.
- Get *probably* $\Theta(N)$ *time* for all k by adapting quicksort:
  - Partition around some pivot, p, as in quicksort, arrange that pivot ends up at dividing line.
  - Suppose that in the result, pivot is at index m, all elements $\leq$ pivot have indicies $\leq m$.
  - If $m = k$, you're done: p is answer.
  - If $m > k$, recursively select k^{th} from left half of sequence.
  - If $m < k$, recursively select $(k - m - 1)^{\text{th}}$ from right half of sequence.

Selection Example

Problem: Find just item #10 in the sorted version of array:

Initial contents:

51	60	21	-4	37	4	49	10	40*	59	0	13	2	39	11	46	31
0																

Looking for #10 to left of pivot 40:

13	31	21	-4	37	4*	11	10	39	2	0	40	59	51	49	46	60
0																

Looking for #6 to right of pivot 4:

-4	0	2	4	37	13	11	10	39	21	31*	40	59	51	49	46	60
				4												

Looking for #1 to right of pivot 31:

-4	0	2	4	21	13	11	10	31	39	37	40	59	51	49	46	60
									9							

Just two elements; just sort and return #1:

-4	0	2	4	21	13	11	10	31	37	39	40	59	51	49	46	60
									9							

Result: 39

Selection Performance

- For this algorithm, if m roughly in middle each time, cost is

$$\begin{aligned} C(N) &= \begin{cases} 1, & \text{if } N = 1, \\ N + C(N/2), & \text{otherwise.} \end{cases} \\ &= N + N/2 + \ldots + 1 \\ &= 2N - 1 \in \Theta(N) \end{aligned}$$

- But in worst case, get $\Theta(N^2)$, as for quicksort.
- By another, non-obvious algorithm, can get $\Theta(N)$ worst-case time for all k (take CS170).