What Are the Questions?

ıcipal concern throughout engineering:

eer is someone who can do for a dime what any fool
r a dollar."

ın

ıal cost (for programs, time to run, space requirements).

ent costs: How much engineering time? When deliv-

ıce costs: Upgrades, bug fixes.

failure: How robust? How safe?

ram *fast enough*? Depends on:

purpose;

input data.

ıace (memory, disk space)?

ends on what input data.

cale, as input gets big?

Cost Measures (Time)

r execution time

o this at home:

e `java FindPrimes 1000`

es: easy to measure, meaning is obvious.

te where time is critical (real-time systems, e.g.).

ages: applies only to specific data set, compiler, ma-

.

tement counts of # of times statements are executed:

es: more general (not sensitive to speed of machine).

ages: doesn't tell you actual time, still applies only to
ata sets.

ecution times:

ormulas for execution times as functions of input size.

es: applies to all inputs, makes scaling clear.

age: practical formula must be approximate, may tell
about actual time.

Handy Tool: Order Notation

try to produce specific functions that specify size, but
ies of functions with similarly behaved magnitudes.

nething like "f is bounded by g if it is in g's family."

tion $g(x)$, the functions $2g(x)$, $0.5g(x)$, or for any $K > 0$,
ave the same "shape". So put all of them into g's family.

ı $h(x)$ such that $h(x) = K \cdot g(x)$ for $x > M$ (for some
has g's shape "except for small values." So put all of
family.

nits, throw in all functions whose absolute value is ev-
some member of g's family. Call this set $O(g)$ or $O(g(n))$.

r limits, throw in all functions whose absolute values is
$\geq$ some member of g's family. Call this set $\Omega(g)$.

ne $\Theta(g) = O(g) \cap \Omega(g)$—the set of functions *bracketed*
by two members of g's family.

CS61B Lecture #16: Complexity

Enlightening Example

a text corpus (say 10^8 bytes or so), and find and print
equently used words, together with counts of how often

nuth): Heavy-Duty data structures

e implementation, randomized placement, pointers ga-
ral pages long.

oug McIlroy): UNIX shell script:

```
'[:alpha:]' '[\n*]' < FILE | \

 \
r -k 1,1 | \
```

ter?

h faster,

ok 5 minutes to write and processes 100MB in ≈ 50 sec.

.

cases, almost anything will do: Keep It Simple.

Asymptotic Cost

ecution time lets us see *shape* of the cost function.

e approximating anyway, pointless to be precise about
ıs:

on small inputs:

ays pre-calculate some results.

for small inputs not usually important.

more interested in *asymptotic behavior* as input size
s very large.

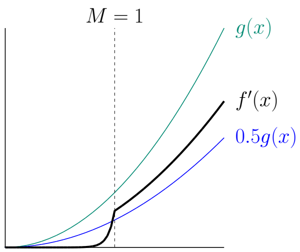
factors (as in "off by factor of 2"):

anging machines causes constant-factor change.

ract away from (i.e., ignore) these things?

Big Omega

y bounding from below:



$\geq \frac{1}{2}g(x)$ as long as $x > 1$,

g's "bounded-below family," written

$$f'(x) \in \Omega(g(x)),$$

gh $f(x) < g(x)$ everywhere.

de: Various Mathematical Pedantry

if I am going to talk about $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$ as sets of really should write, for example,

$f \in O(g)$ instead of $f(x) \in O(g(x))$

$(x) \in O(g(x))$ is short for $\lambda\, x.\ f(x) \in O(\lambda\, x.\ g(x))$.

d notation outside this course, in fact, is $f(x) = O(g(x))$, ly, I think that's a serious abuse of notation.

Why It Matters

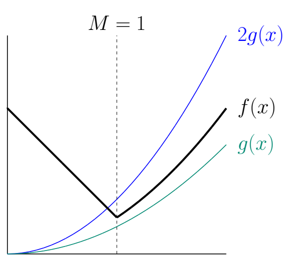
ientists often talk as if constant factors didn't matter e difference of $\Theta(N)$ vs. $\Theta(N^2)$.

ey do matter, but at some point, constants always get

$\sqrt{n}$	n	$n \lg n$	n^2	n^3	2^n
1.4	2	2	4	8	4
2	4	8	16	64	16
2.8	8	24	64	512	256
4	16	64	256	4,096	65,636
5.7	32	160	1024	32,768	4.2×10^9
8	64	384	4,096	262,144	1.8×10^{19}
11	128	896	16,384	2.1×10^9	3.4×10^{38}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
32	1,024	10,240	1.0×10^6	1.1×10^9	1.8×10^{308}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1024	1.0×10^6	2.1×10^7	1.1×10^{12}	1.2×10^{18}	$6.7 \times 10^{315,652}$

Big Oh

y bounding from above.



$2g(x)$ as long as $x > 1$,

g's "bounded-above family," written

$$f(x) \in O(g(x)),$$

gh (in this case) $f(x) > g(x)$ everywhere.

Big Theta

previous slides, we not only have $f(x) \in O(g(x))$ and $)),\ldots$

$(x) \in \Omega(g(x))$ and $f'(x) \in O(g(x))$.

marize this all by saying $f(x) \in \Theta(g(x))$ and $f'(x) \in$

How We Use Order Notation

mathematics, you'll see $O(\ldots)$, etc., used generally to ds on functions.

$$\pi(N) = \Theta(\frac{N}{\ln N})$$

d prefer to write

$$\pi(N) \in \Theta(\frac{N}{\ln N})$$

is the number of primes less than or equal to N.)

e things like

$= x^3 + x^2 + O(x)$ (or $f(x) \in x^4 + x^2 + O(x)$),

$f(x) = x^3 + x^2 + g(x)$ where $g(x) \in O(x)$.

oses, the functions we will be bounding will be *cost functions* that measure the amount of execution time or the ace required by a program or algorithm.

Using the Notation

order notation for any kind of real-valued function.

hem to describe cost functions. Example:

```
 position of X in list L, or -1 if not found. */
(List L, Object X) {

: = 0; L != null; L = L.next, c += 1)
 (X.equals(L.head)) return c;
n -1;
```

esentative operation: number of `.equals` tests.

th of L, then loop does *at most* N tests: *worst-case* sts.

al # of instructions executed is roughly proportional worst case, so can also say worst-case time is $O(N)$, f units used to measure.

provision (in defn. of $O(\cdot)$) to ignore empty list.

Effect of Nested Loops

s often lead to polynomial bounds:

```
 i = 0; i < A.length; i += 1)
nt j = 0; j < A.length; j += 1)
 (i != j && A[i] == A[j])
 return true;
lse;
```

: is $O(N^2)$, where $N =$ `A.length`. *Worst-case time* is

icient though:

```
 i = 0; i < A.length; i += 1)
nt j = i+1; j < A.length; j += 1)
 (A[i] == A[j]) return true;
lse;
```

ase time is proportional to

$-1+N-2+\ldots+1 = N(N-1)/2 \in \Theta(N^2)$

ic time unchanged by the constant factor).

Binary Search: Slow Growth

```
 is an element of S[L .. U]. Assumes
ding order, 0 <= L <= U-1 < S.length. */
tring X, String[] S, int L, int U) {
eturn false;
)/2;
 X.compareTo(S[M]);
 0) return isIn(X, S, L, M-1);
ect > 0) return isIn(X, S, M+1, U);
 true;
```

case time, $C(D)$, (as measured by # of calls to `.compareTo`), ize $D = U - L + 1$.

S[M] from consideration each time and look at half the $D = 2^k - 1$ for simplicity, so:

$$\begin{aligned} C(D) &= \begin{cases} 0, & \text{if } D \le 0, \\ 1 + C((D-1)/2), & \text{if } D > 0. \end{cases} \\ &= \underbrace{1+1+\ldots+1}_{k}+0 \\ &= k = \lg(D+1) \in \Theta(\lg D) \end{aligned}$$

ne Intuition on Meaning of Growth

oblem can you solve in a given time?

ving table, left column shows time in microseconds to problem as a function of problem size N.

the *size of problem* that can be solved in a second, (31 days), and century, for various relationships be- equired and problem size.

size.

) for ze N	Max N Possible in 1 second	1 hour	1 month	1 century
	10^{300000}	$10^{1000000000}$	$10^{8\cdot 10^{11}}$	$10^{10^{14}}$
	10^6	$3.6\cdot 10^9$	$2.7\cdot 10^{12}$	$3.2\cdot 10^{15}$
N	63000	$1.3\cdot 10^8$	$7.4\cdot 10^{10}$	$6.9\cdot 10^{13}$
	1000	60000	$1.6\cdot 10^6$	$5.6\cdot 10^7$
	100	1500	14000	150000
	20	32	41	51

Be Careful

: that the worst-case time is $O(N^2)$, since $N \in O(N^2)$ bounds are loose.

ase time is $\Omega(N)$, since $N \in \Omega(N)$, but that does *not* e loop *always* takes time N, or even $K \cdot N$ for some K.

are just saying something about the *function* that maps *argest possible* time required to process any array of

ch as possible about our worst-case time, we should try ound: in this case, we can: $\Theta(N)$.

hat still tells us nothing about *best-case* time, which n we find X at the beginning of the loop. Best-case time

rsion and Recurrences: Fast Growth

: of recursion. In the worst case, both recursive calls

```
ff X is a substring of S */
curs(String S, String X) {
uals(X)) return true;
ngth() <= X.length()) return false;

(S.substring(1), X) ||
(S.substring(0, S.length()-1), X);
```

) to be the worst-case cost of `occurs(S,X)` for S of f fixed size N_0, measured in # of calls to `occurs`. Then

$$C(N) = \begin{cases} 1, & \text{if } N \le N_0, \\ 2C(N-1)+1 & \text{if } N > N_0 \end{cases}$$

ws exponentially:

$N-1)+1 = 2(2C(N-2)+1)+1 = \ldots = \underbrace{2(\cdots 2}_{N-N_0}\cdot 1+1)+\ldots+1$

$^{N_0} + 2^{N-N_0-1} + 2^{N-N_0-2} + \ldots + 1 = 2^{N-N_0+1} - 1 \in \Theta(2^N)$

...other Typical Pattern: Merge Sort

```
... L) {
...() < 2) return L;
... and L1 of about equal size;
...));  L1 = sort(L1);
...e of L0 and L1
```

Merge ("combine into a single ordered list") takes time proportional to size of its result.

...at size of L is $N = 2^k$, worst-case cost function, $C(N)$,
... merge time (which is proportional to # items merged):

$$
\begin{aligned}
C(N) &= \begin{cases} 0, & \text{if } N < 2; \\ 2C(N/2) + N, & \text{if } N \geq 2. \end{cases} \\
&= 2(2C(N/4) + N/2) + N \\
&= 4C(N/4) + N + N \\
&= 8C(N/8) + N + N + N \\
&= N \cdot 0 + \underbrace{N + N + \ldots + N}_{k = \lg N} \\
&= N \lg N
\end{aligned}
$$

...an say it's $\Theta(N \lg N)$ for arbitrary N (not just 2^k).