Lecture #2: Functions, Expressions, Environments

Announcements

- HW 1 available. Due next Thursday.
- Piazza post @30 contains an index of useful Piazza threads and Zoom links (such as orientation links).
- Come to the weekly exam prep sections, Fridays 3:10-4:30 (right after Friday lecture). This first week: how exams are structured, studying advice, and our tips for succeeding in the course. Next week we will dive into solving exam level problems.
- DSP students should submit their accommodation letters ASAP.

From Last Time

- From last lecture: *Values* are data we want to manipulate and in particular,
- *Functions* are values that perform computations on values.
- *Expressions* denote computations that produce values.
- Today, we'll look in some detail at how functions operate on data values and how expressions denote these operations.
- Why start with functions? Ultimately, function calls do all the work of computation in Python, together with assignments.
- Many constructs in Python that one may also think of as basic are actually "*syntactic sugar*" for function calls. Example:

  ```
  (3 + 7 + 10) * (1000 - 8) / 992 - 17
  ```

  is equivalent (ignoring some necessary imports) to

  ```
  sub(truediv(mul(add(add(3, 7), 10), sub(1000, 8)), 992), 17)
  ```

- As usual, although our concrete examples all involve Python, the actual concepts apply almost universally to programming languages.

Functions as Values

- In grade school, we tend to think of mathematics as having to do with numbers—values are numeric.
- But in mathematics, there are many other kinds of values and standard operations on them.
- For example, the derivative ($\frac{\mathrm{d}}{\mathrm{dx}}$) and integral ($\int$) operators operate on functions to produce new ones.
- And there are sets of functions and sequences of functions as well.
- Originally, functions were not typically treated as full-fledged (or "first class") values in programming languages, but this has changed over the years.
- Python's functions *are* first-class values, and may be assigned to variables, passed to and returned from functions, and stored in data structures.

Functions Values

- For this lecture, we're going to use the Python tutor's notation for function values:

  func abs(number), func add(left, right)

- The above are *primitive* (or *native*) function values, provided by Python.
- New functions result from evaluating *function definitions* such as

```
def saxb(a, x, b):    # Header: Name and formal parameters
    return a * x + b  # Body: Computation performed by function
```

  which creates a function value we'll write in this lecture as

  func saxb(a, x, b): return a * x + b

- The green parenthesized lists indicate the number of *parameter values* or *inputs* the functions operate on (this information is also known as a function's *signature*).

Functions (II)

- For intrinsic functions and those created by `def`, Python actually maintains an *intrinsic name* (such as the saxb in "func saxb(a, x, b).")
- But often in mathematics, functions are *anonymous,* as in "the function that takes three real number, a, x, and b and yields $ax + b$."
- A traditional mathematical notation for such anonymous functions looks like this:

  $$\lambda\, a, x, b.\, ax + b$$

  (that's the Greek letter lambda).
- We'll write these values as

  func λ(a, x, b)

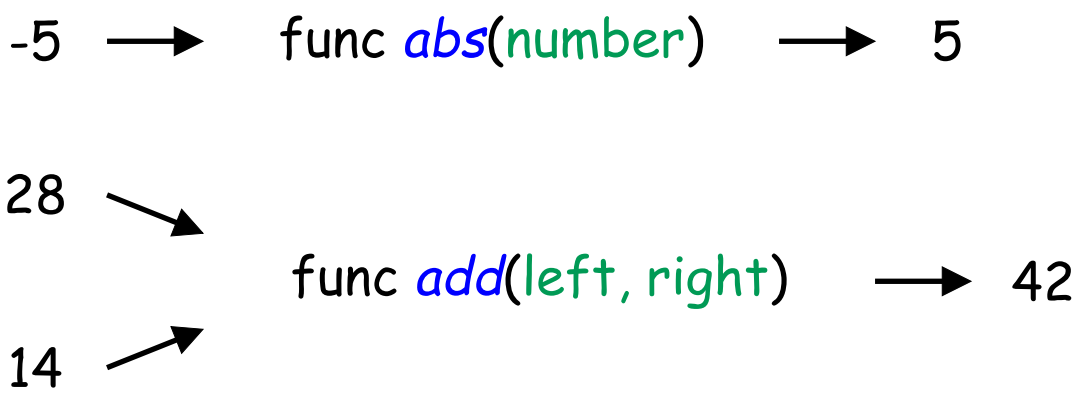
  as the Python tutor does, or, when emphasizing the body, as

  func λ(a, x, b): a * x + b
- These anonymous functions are created by Python *lambda expressions* such as:

```
lambda a, x, b: a * x + b
```

Pure Functions

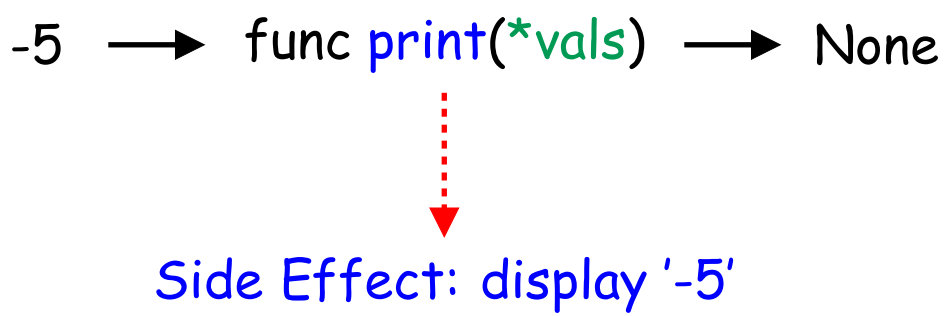
- The fundamental operation on function values is to *call* or *invoke* them, which means giving them one value for each parameter they expect and having them produce the result of their computation on these values:



- These two functions are *pure:* their output depends only on their input parameters' values, and they do nothing in response to a call but compute a value.

Impure Functions

- Functions may do additional things when called besides returning a value.
- We call such things *side effects.*
- Example: the built-in print function:



- Displaying text is print's side effect. Its value, in fact, is generally useless (it always returns the special value None).
- (The notation *vals is Python's way of designating an arbitrary number (0 or more) of parameters.)

Other Kinds of Impurity

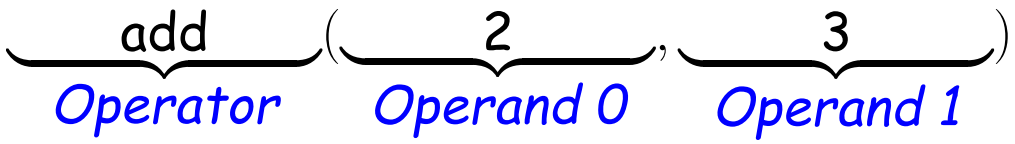
- Most side-effects involve changing the value of some variable.
- Example: the function random.randint:

```
>>> from random import randint
>>> randint(0, 100)   # Random number in 0--100.
13
>>> randint(0, 100)   # Different result: Something must have changed!
55
```

- You can deduce from this behavior that there must be some variable hidden away somewhere that changes every time `randint` is called.
- In fact, you can think of printing from the last slide as a change to another invisible variable (containing the "stuff that's been printed").
- We use the hidden-variable idea to describe side effects when doing formal mathematical descriptions of programming languages.

Call Expressions

- A call expression denotes the operation of calling a function.
- Consider add(2, 3):



- The operator and the operands are all themselves expressions (recursion again).
- To evaluate this call expression:
  - Evaluate the operator (let's call the resulting value C);
  - Evaluate the operands in the order they appear (let's call the resulting values P_0 and P_1)
  - Call C (which must be a function) with parameters P_0 and P_1.
- Together with the definitions for base cases (mostly literal expressions and symbolic names), this describes how to evaluate any call.

Example: From Expression to Value (I)

Let's evaluate the following expression (`0x` and `0o` are Python's way of saying "base 16" and "base 8." I've used them to contrast expressions and values.):

mul(add(2, mul(0x10, 0o10)), add(0x3, 5))

In the following sequence, values (as opposed to expressions) are shown in boxes to indicate that they need no further evaluation.

mul(add(2, mul(0x10, 0o10)), add(0x3, 5))

func mul(x, y)(add(2, mul(0x10, 0o10)), add(0x3, 5))

func mul(x, y)(func add(x, y)(2, mul(0x10, 0o10)), add(0x3, 5))

func mul(x, y)(func add(x, y)(2, mul(0x10, 0o10)), add(0x3, 5))

func mul(x, y)(func add(x, y)(2, func mul(x, y)(0x10, 0o10)), add(0x3, 5))

func mul(x, y)(func add(x, y)(2, func mul(x, y)(16, 0o10)), add(0x3, 5))

func mul(x, y)(func add(x, y)(2, func mul(x, y)(16, 8)), add(0x3, 5))

func mul(x, y)(func add(x, y)(2, 128), add(0x3, 5))

Example: From Expression to Value (II)

func mul(x, y)(func add(x, y)(2, 128), add(0x3, 5))
func mul(x, y)(130, add(0x3, 5))
func mul(x, y)(130, func add(x, y)(0x3, 5))
func mul(x, y)(130, func add(x, y)(3, 5))
func mul(x, y)(130, func add(x, y)(3, 5))
func mul(x, y)(130, 8)
1040

Example: Print

What about an expression with side effects? (Skipping some steps for brevity)

1. print(print(1), print(2))
2. func print(*vals)(func print(*vals)(1), print(2))
3. func print(*vals)(None , print(2))
   and print '1'.
4. func print(*vals)(None , func print(*vals)(2))
5. func print(*vals)(None , None))
   and print '2'.
6. None
   and print 'None None'.

```
>>> print(print(1), print(2))
1
2
None None
```

Names

- Evaluating expressions that are literals is easy: the literal's text gives all the information needed.
- But how did I evaluate names like add, mul, or print?
- Deduction: there must be another source of information.
- We'll first try a simple approach to describing how to handle the evaluation of names: *substitution* of values for names.
- This won't cover all the cases, however, and so we'll introduce the concept of an *environment*.

Substitution

- Python provides several ways to define names: *assignments* of values to names, *function definitions*, and *parameter passing* to functions.
- Let's try to explain the effect of

```
x = 3
y = x * 2
z = y ** x
```

  by treating each assignment (=) as a *definition*.

- Thus, we get

```
x = 3          x = 3          x = 3          x = 3
y = x * 2      y = 3 * 2      y = 6          y = 6
z = y ** x     z = y ** 3     z = 6 ** 3     z = 216
```

- That is, we *replace names by their definitions (values).*

Parameter Substitution and Functions

- Now consider a simple function definition:

```
def compute(x, y):
    return (x * y) ** x
compute(3, 2)
```

- A **def** statement is sort of like an assignment, but specialized to functional values.
- The **def** statement above defines compute to be "the function of x and y that returns $(xy)^x$," or as we have written it before:

  func compute(x, y): return (x * y) ** x

- So evaluation of `compute(3, 2)`, as described previously, eventually gives us

  func compute(x, y): return (x * y) ** x (3 , 2)

- Now what? How do these calls on user-defined functions work once the operands are evaluated?

Substitution and Formal Parameters

- Evaluating a function call such as

  func compute(x, y): return (x * y) ** x (3 , 2)

  from the last slide is like a *simultaneous assignment* to or substitution for x and y—the *formal parameters* of `compute`.

- That is, we replace the whole expression with

  return (3 * 2) ** 3

  and (eventually), just 216 .

- (Let's just ignore the pesky **return**: it's basically Python syntax to indicate which value the function produces.)

Getting Fancy

- What about this?

```
def incr(n):
    def f(x):
        return n + x
    return f

incr(5)(6)
```

- The incr function returns a function (f), not a number. We then call this function on 6.
- What happens?

Answer (Part I)

- First, deal with incr:

```
def incr(n):
    def f(x):
        return n + x
    return f
incr(5)(6)
```

  which first defines `incr` to be the value:

  func incr(n): def f...return f

- So when we evaluate `incr(5)`, we end up calling

  func incr(n): def f...return f (5)

  and substitution gives

```
def f(x):
    return 5 + x
return f
```

- That gives us a value for `incr(5)` of

  func f(x): return 5 + x

Answer (Part II)

- So now (after evaluating the argument 6), we evaluate

  func f(x): return 5 + x (6)

- resulting in the evaluation of

  5 + 6

- Finally giving 11

Complication

- What do we get out of

```
def hmmmm(x):
    def f(x):
        return x
    return f
hmmmm(5)(6)
```

- Does this give us 5 or 6?
- This boils down to whether `hmmmm(5)` gives

  func f(x): def f(x): return 5

  or

  func f(x): def f(x): return x

- We opt for the second one.

Free vs. Bound, Hiding

- Again, given

```
def hmmmm(x):
    def f(x):
        return x
    return f
hmmmm(5)(6)
```

- The inner definition (def f) "protects" the `x` in **return** `x` from substitution when calling `hmmmm`.
- Formally, we say that the substitution caused by `hmmmm(5)` replaces *free occurrences* of `x` in the body of `hmmmm` only.
- A free occurrence of a name in a statement or expression is one that is not defined (*bound*) within that statement.
- Since the `x` in the **return** statement is defined to be the formal parameter of `f`, it is bound, and therefore not replaced.
- Another way to say it is that the inner definition of `x` (in `f`) *hides* the outer one (in `hmmmm`) within the body of `f`.

Trouble

- Alas, even besides these complications, the substitution approach doesn't entirely work.
- Example:

```
x = 4
x = 8
print(x)
```

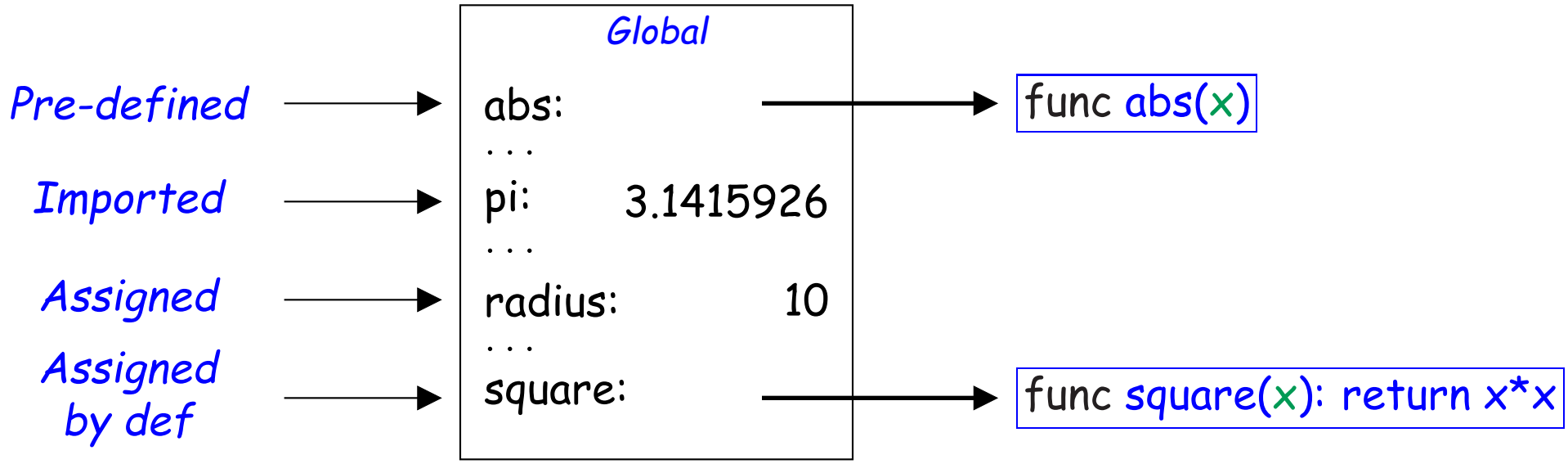
- If we just substitute for the first x as before:

```
x = 4
x = 8    # or even worse: 4 = 8
print(4)
```

- . . . we get a wrong result (4 instead of 8).
- After one substitution, x isn't around any more to substitute for.
- We need a more comprehensive answer to the question of how function calls work, one that takes multiple assignments to variables into account.

New Explanation: Environments

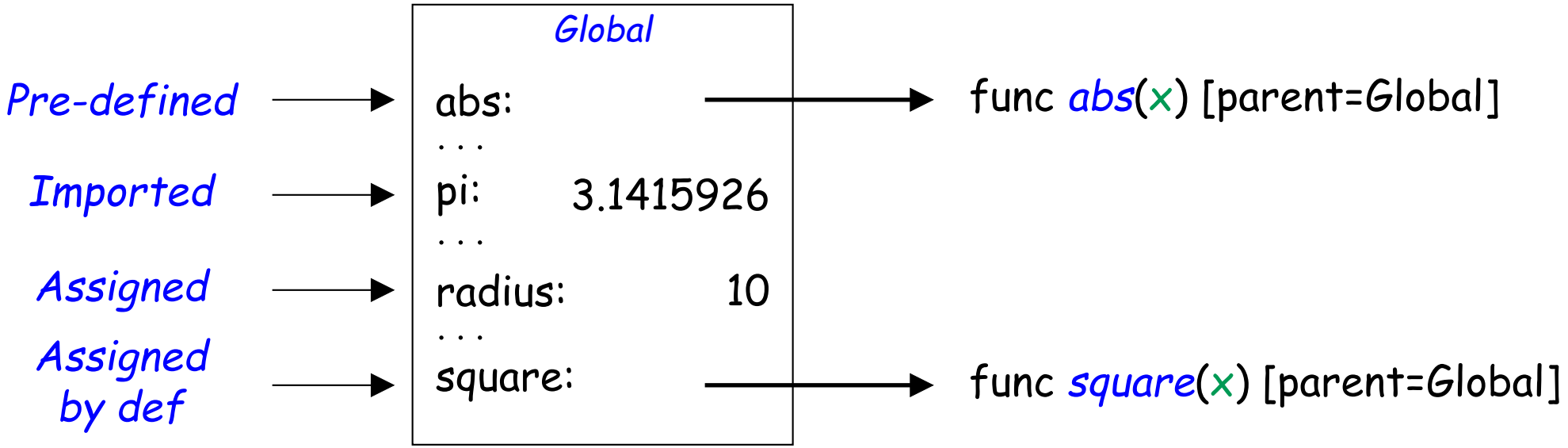
- An environment is a mapping from names to values.
- We say that a name is *bound to* a value in this environment.
- In its simplest form, it consists of a single *global environment frame*:




```
from math import pi
radius = 10
def square(x): return x * x
```

Diagrams in Python Tutor

- You'll be using the Python Tutor from time to time, which uses a somewhat different notation for function values. Might as well get used to it (we'll explain the "parent=" stuff in a later lecture):




```
from math import pi
radius = 10
def square(x): return x**2
```

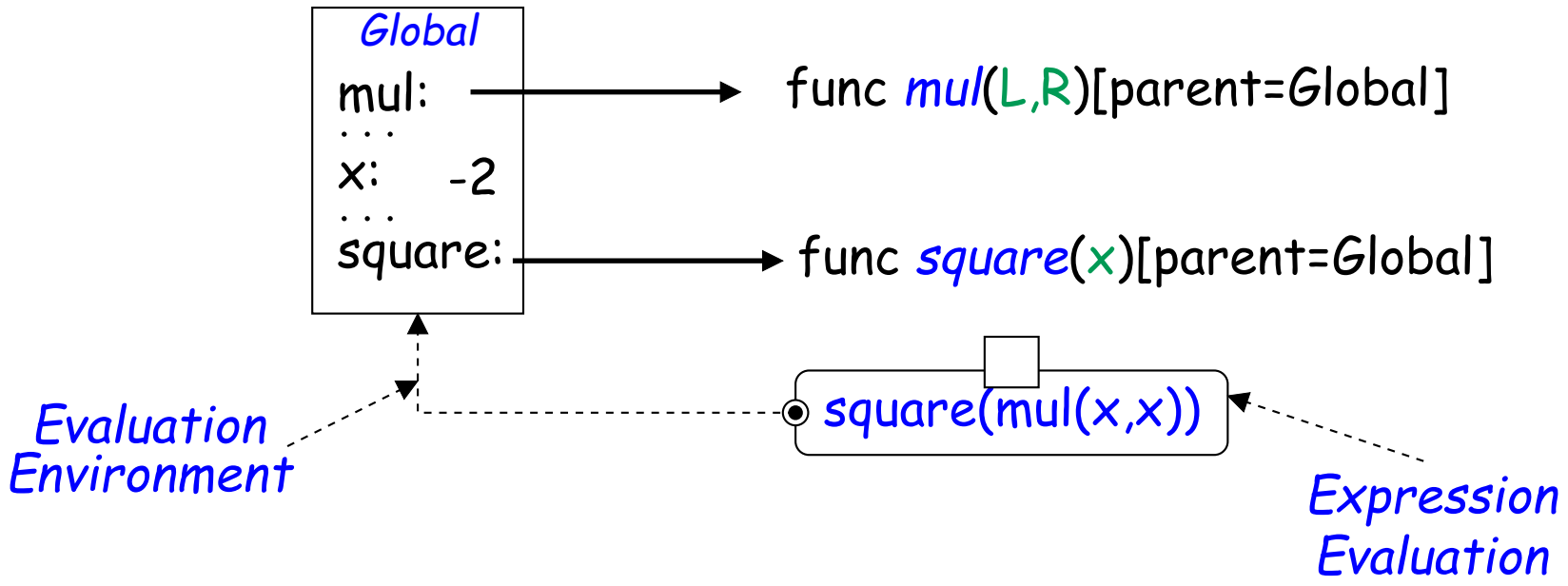
Environments and Evaluation

- Every expression is evaluated in an environment, which supplies the meanings of any names in it.
- Evaluating an expression typically involves first evaluating its subexpressions (the operators and operands of calls, the operands of conventional expressions such as `x*(y+z)`, . . .).
- These subexpressions are evaluated in the same environment as the expression that contains them.
- Once their subexpressions (operator + operands) are evaluated, calls to user-defined functions must evaluate the expressions and statements from the definitions of those functions.

Evaluating User-Defined Function Calls

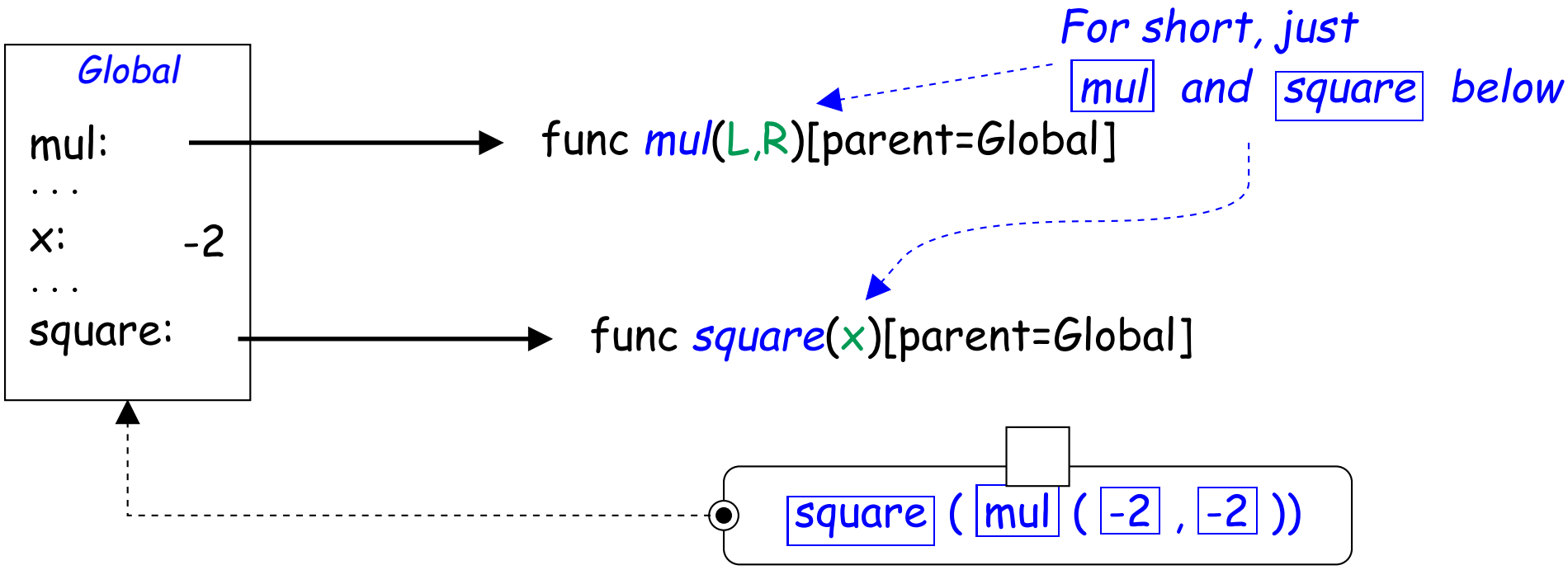
- Consider the expression square(mul(x, x)) after executing

```
from operator import mul
def square(x):
    return mul(x,x)
x = -2
```



Evaluating User-Defined Function Calls (II)

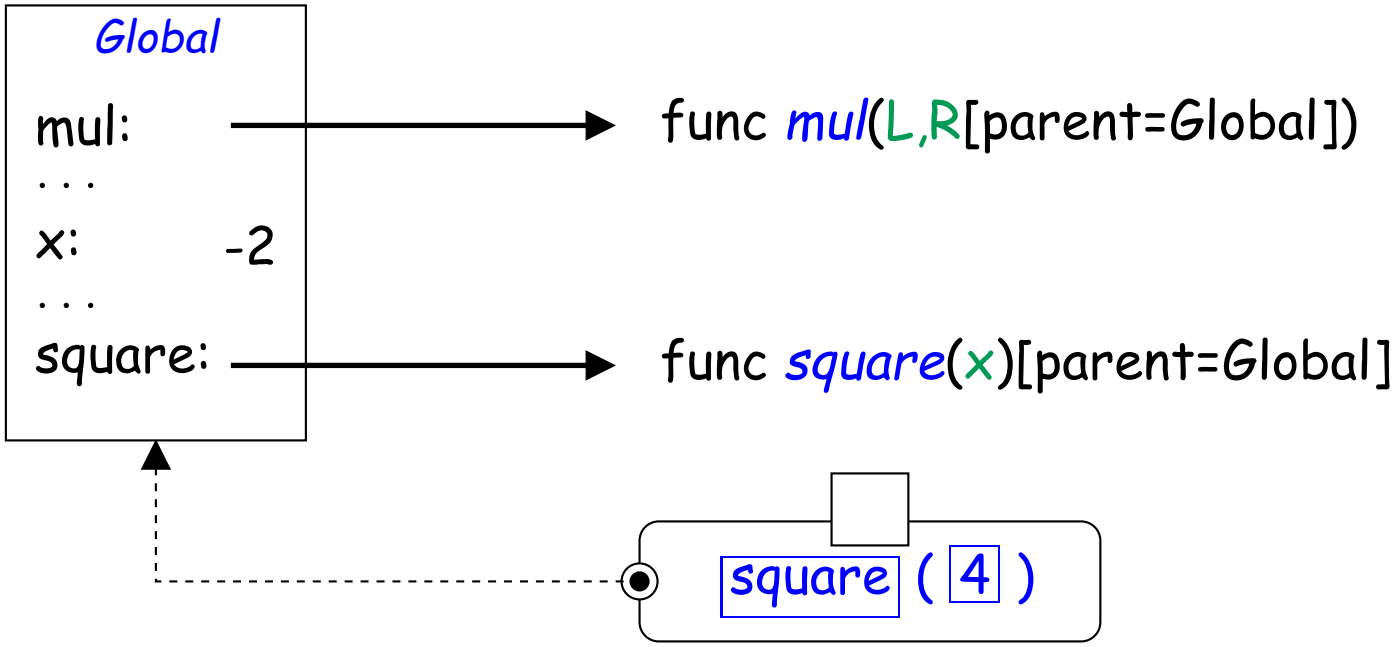
- First evaluate the subexpressions of square(mul(x, x)) in the global environment:



- Evaluating subexpressions x, mul, and square take values from the expression's environment.

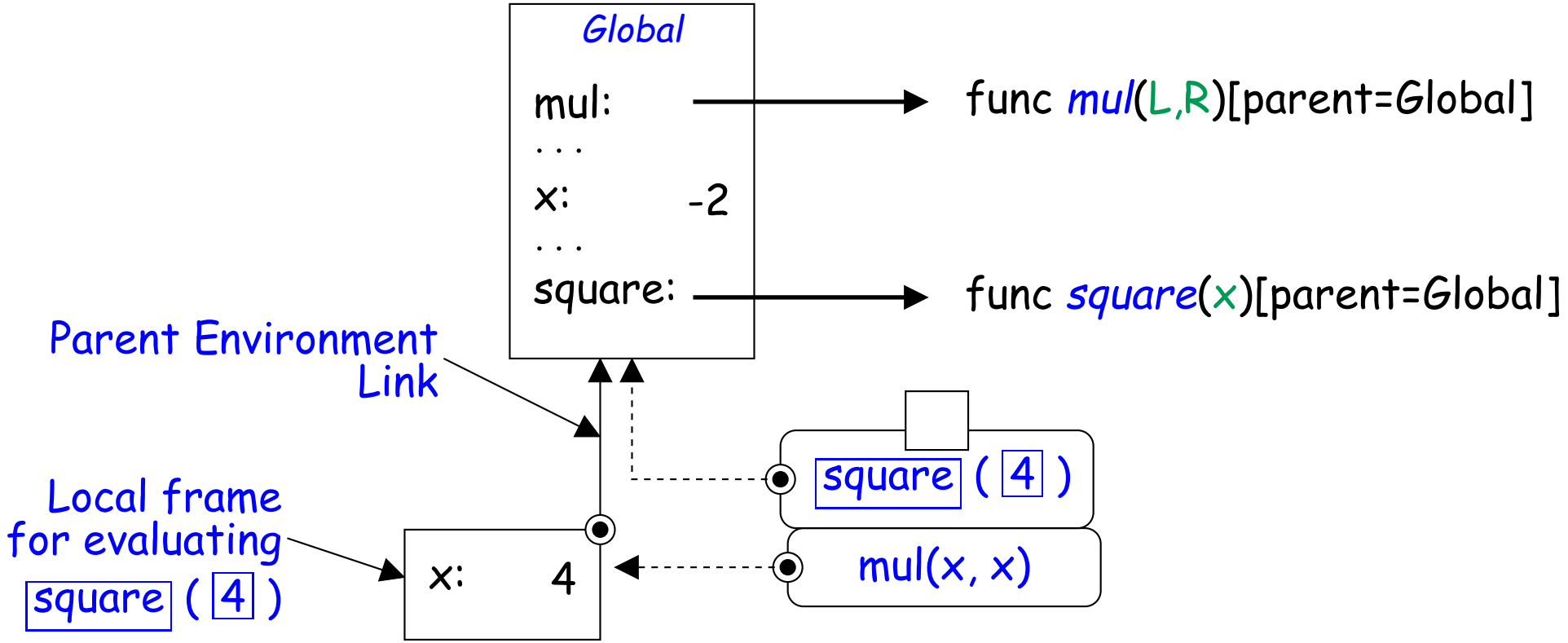
Evaluating User-Defined Functions Calls (III)

- Then perform the primitive multiply function:



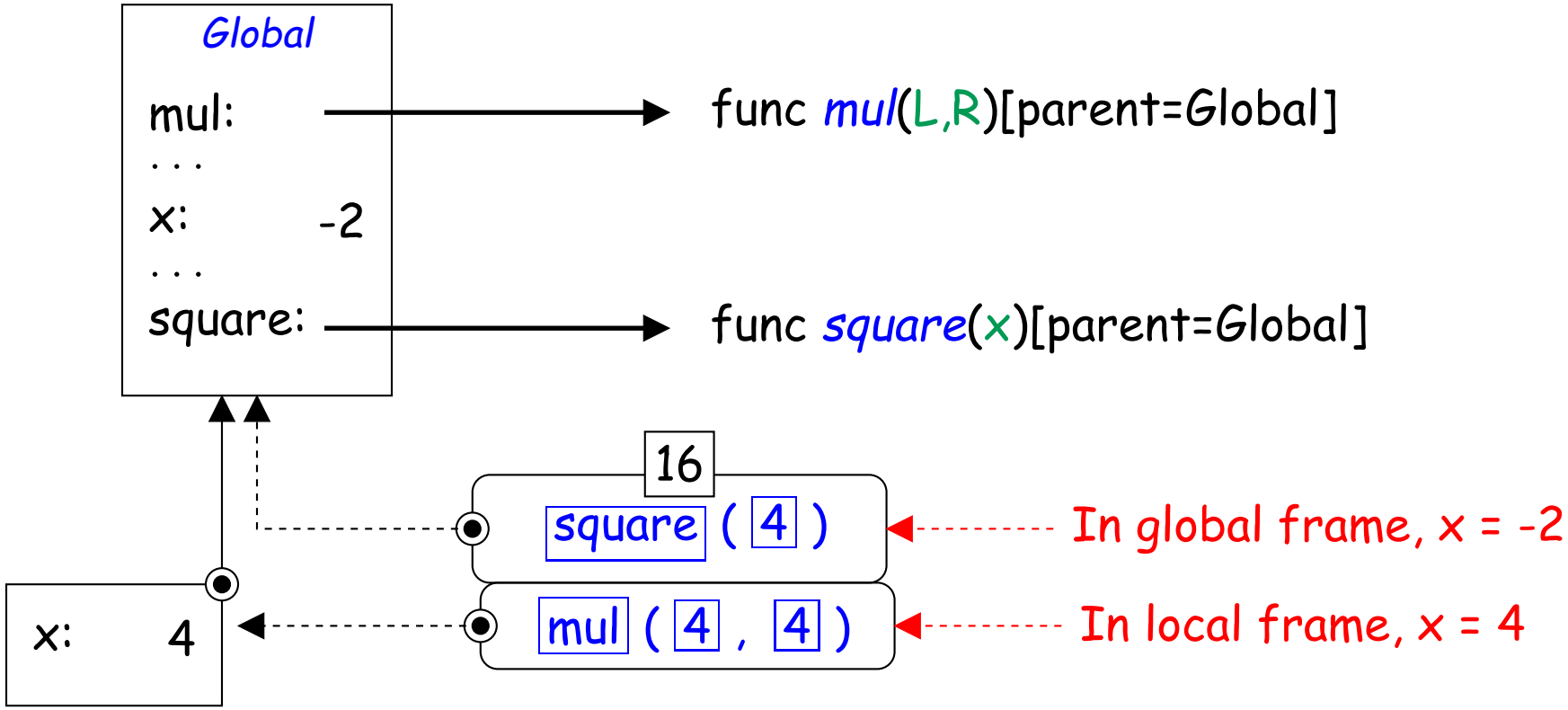
Evaluating User-Defined Functions Calls (IV)

- To explain the parameters to the user-defined square function, extend the environment with a *local environment frame*, attached to the frame in which square was defined (the global one in this case), and giving x the operand value.
- Now replace original call with evaluating the body of square in the new local environment.



Evaluating User-Defined Functions Calls (V)

- When we evaluate mul(x, x) in this new environment, we get the same value as before for mul, but the local value for x.
- When evaluating an identifier in a chain of environments, follow the parent environment links to the first frame containing its definition.



So How Does This Help?

- The original problem that led to this whole environment diagram thing was how to deal with:

```
x = 4
x = 8
print(x)
```

- Now it's easy. Each time we assign to *x*, we create a new binding for it in the current evaluation frame (replacing the old one, if any).
- We get the new (last assigned) value when we look up *x* in the modified environment.
- The complication of hiding is also easy to explain: to find the meaning of a name, we follow a chain of environment frames and stop at the first one with the desired definition.