Efficiency

Announcements

Measuring Efficiency

Recursive Computation of the Fibonacci Sequence

Our first example of tree recursion:

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http://en.wikipedia.org/wiki/File:Fibonacci.jpg

Recursive Computation of the Fibonacci Sequence

Our first example of tree recursion:

fib(5)

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fib(3)
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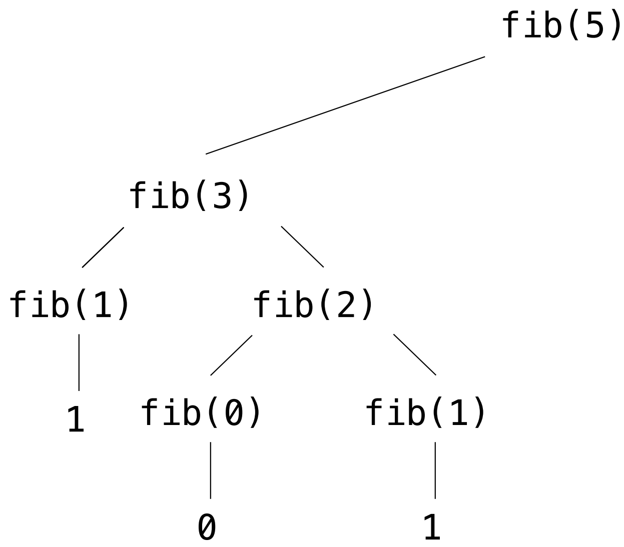


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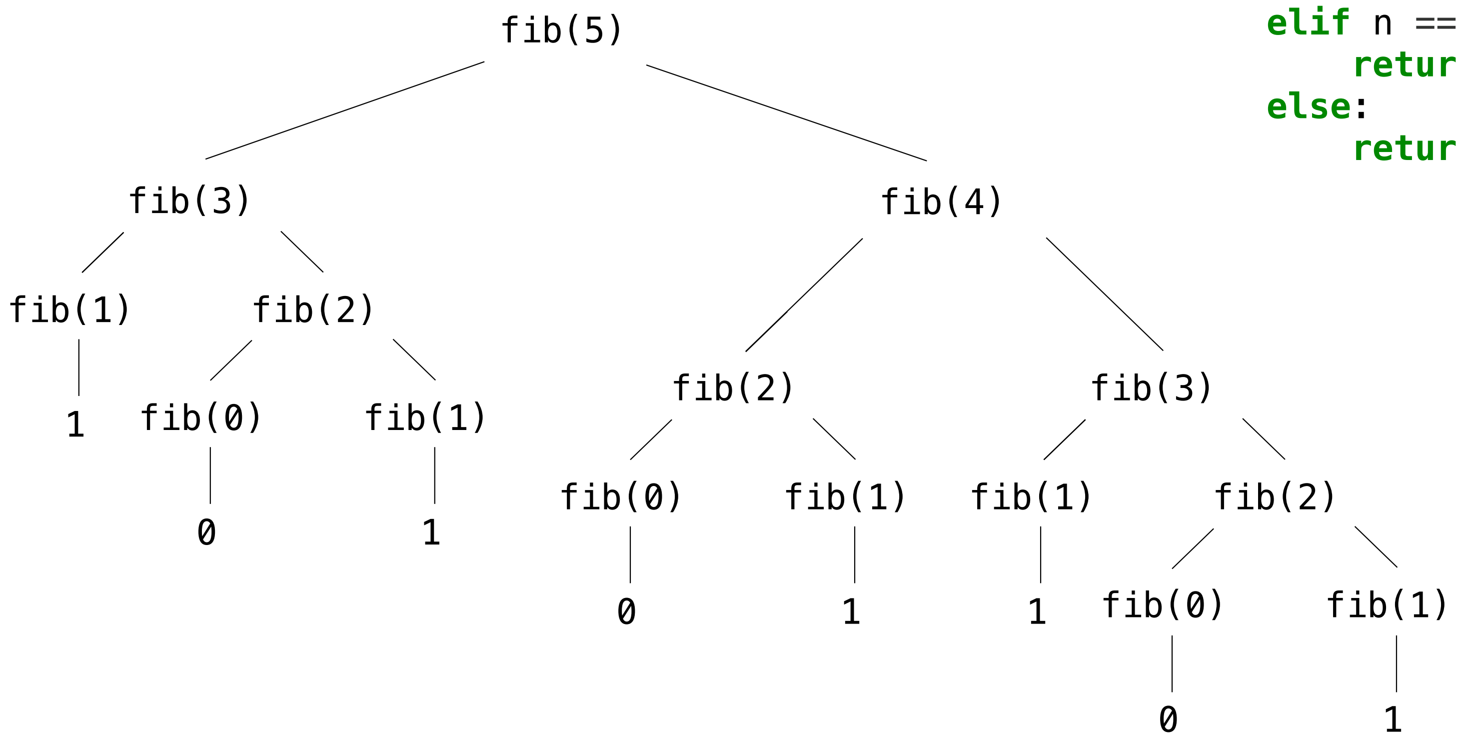

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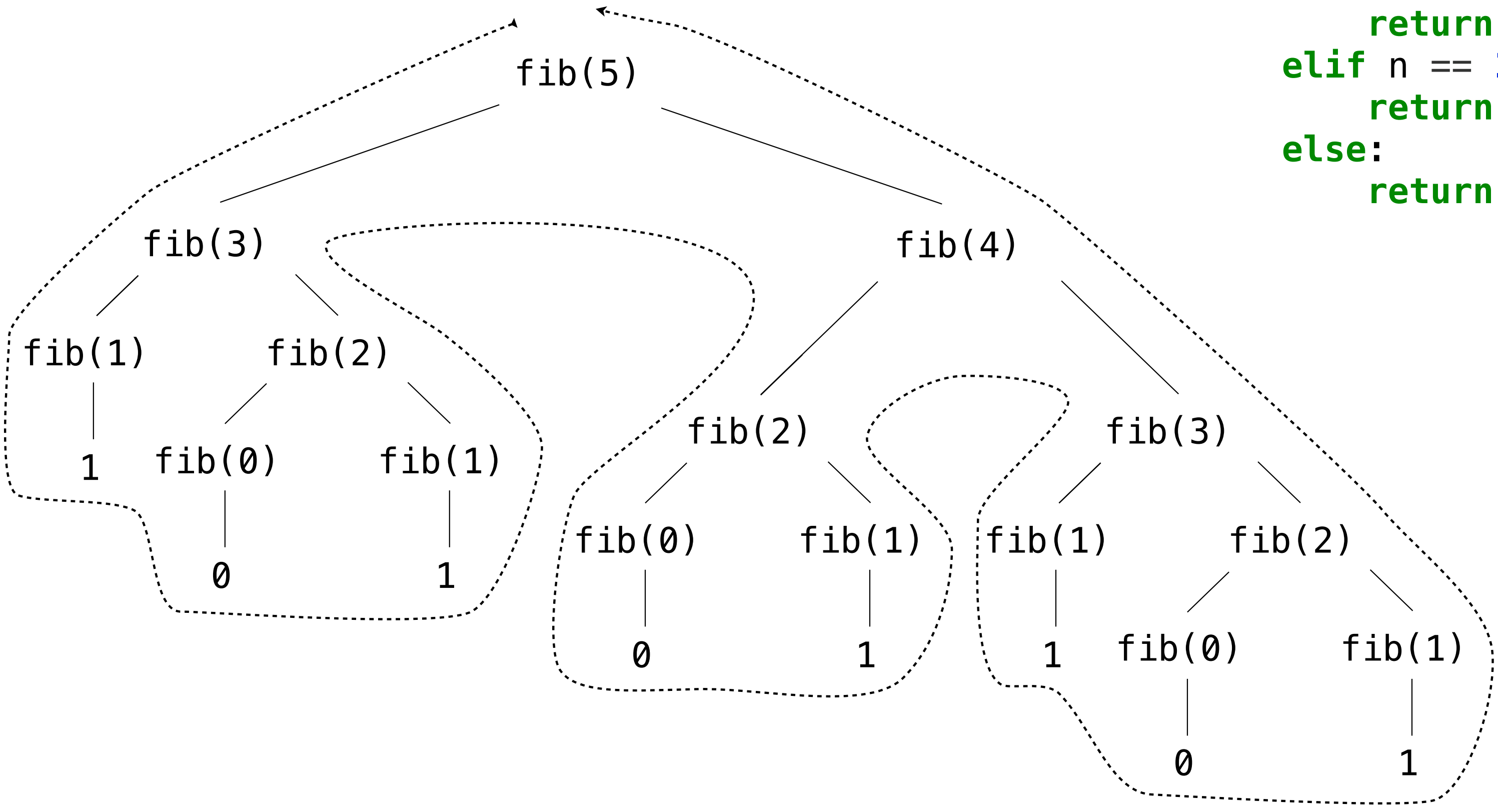


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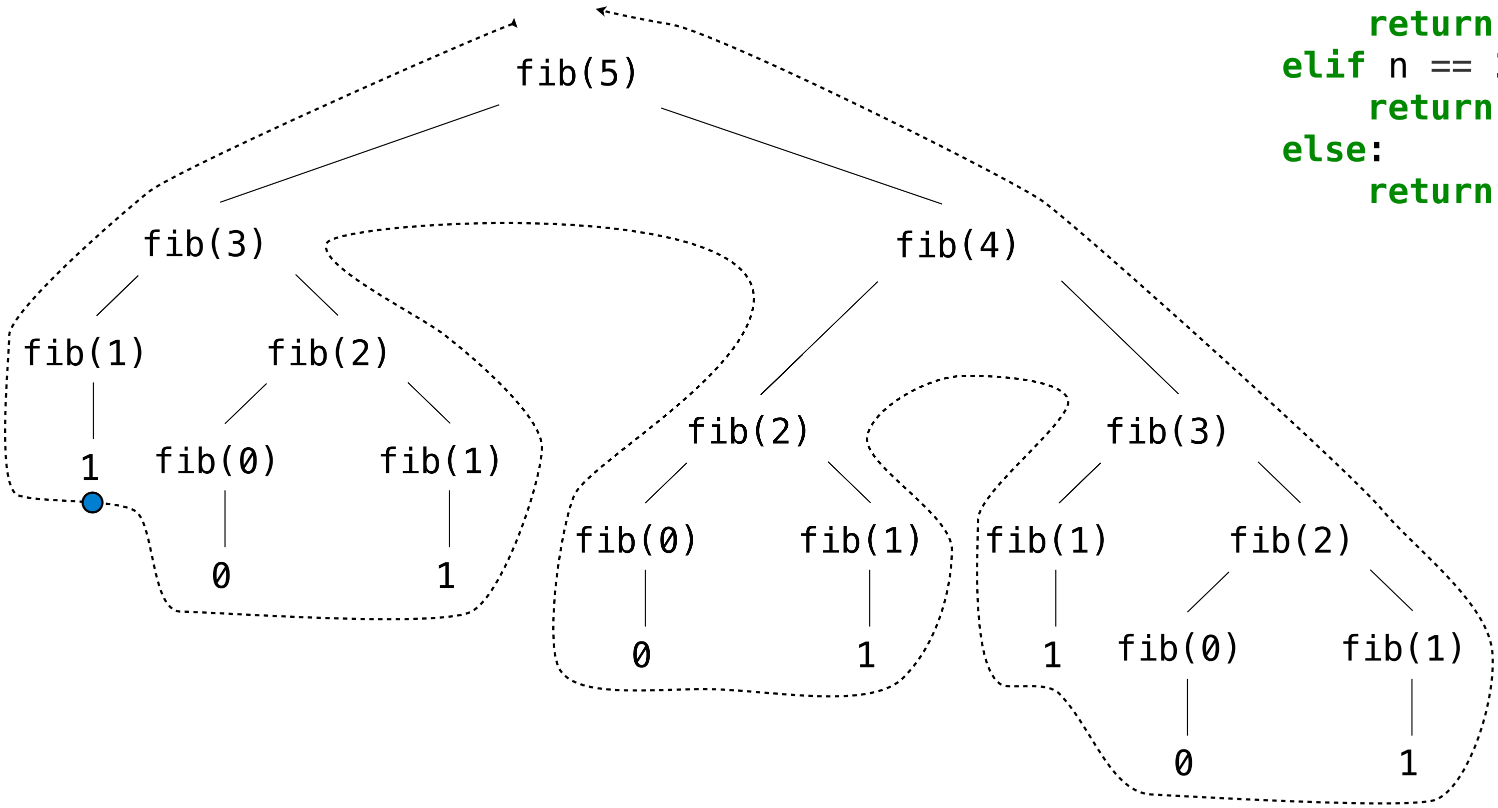

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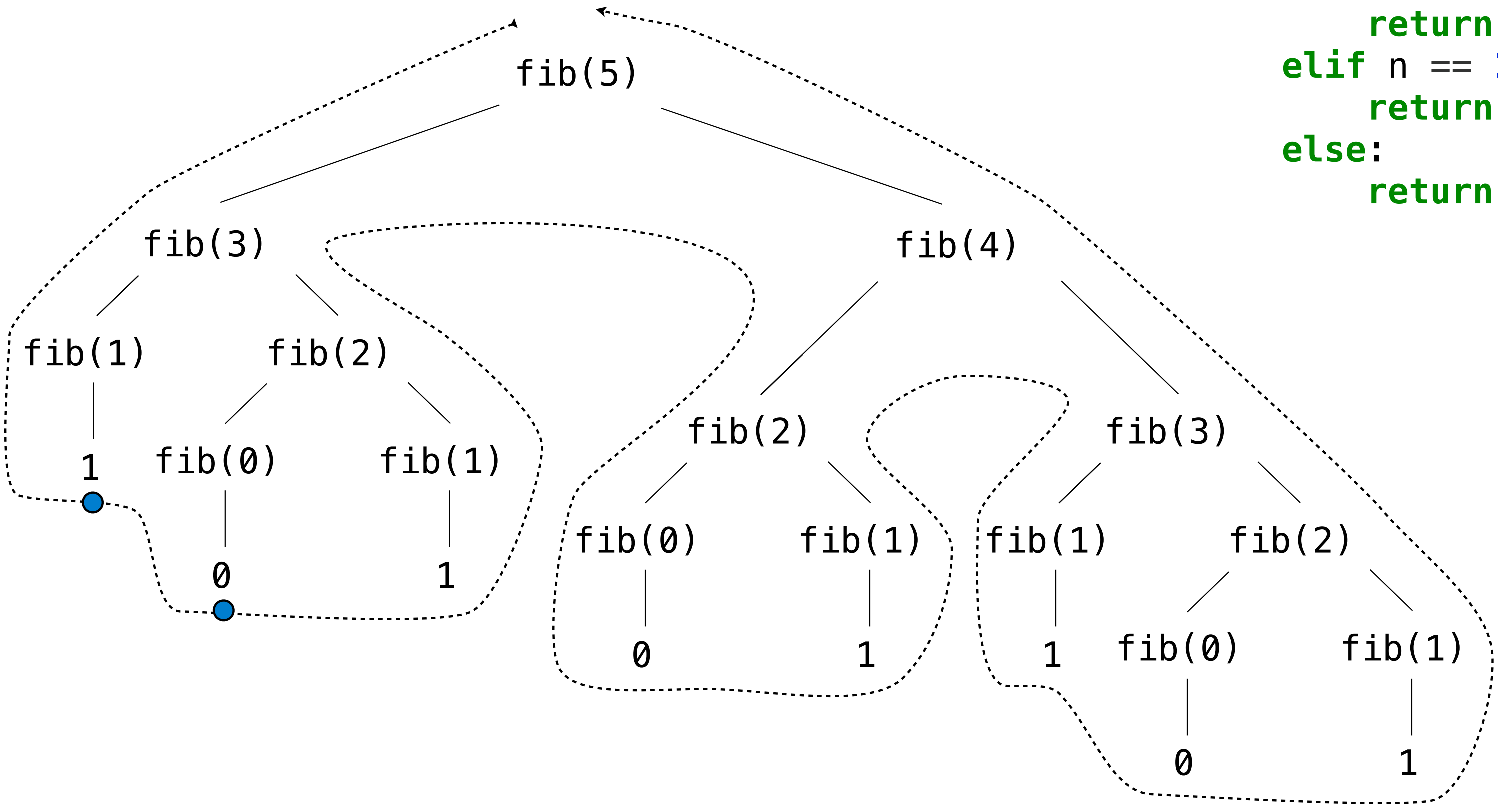

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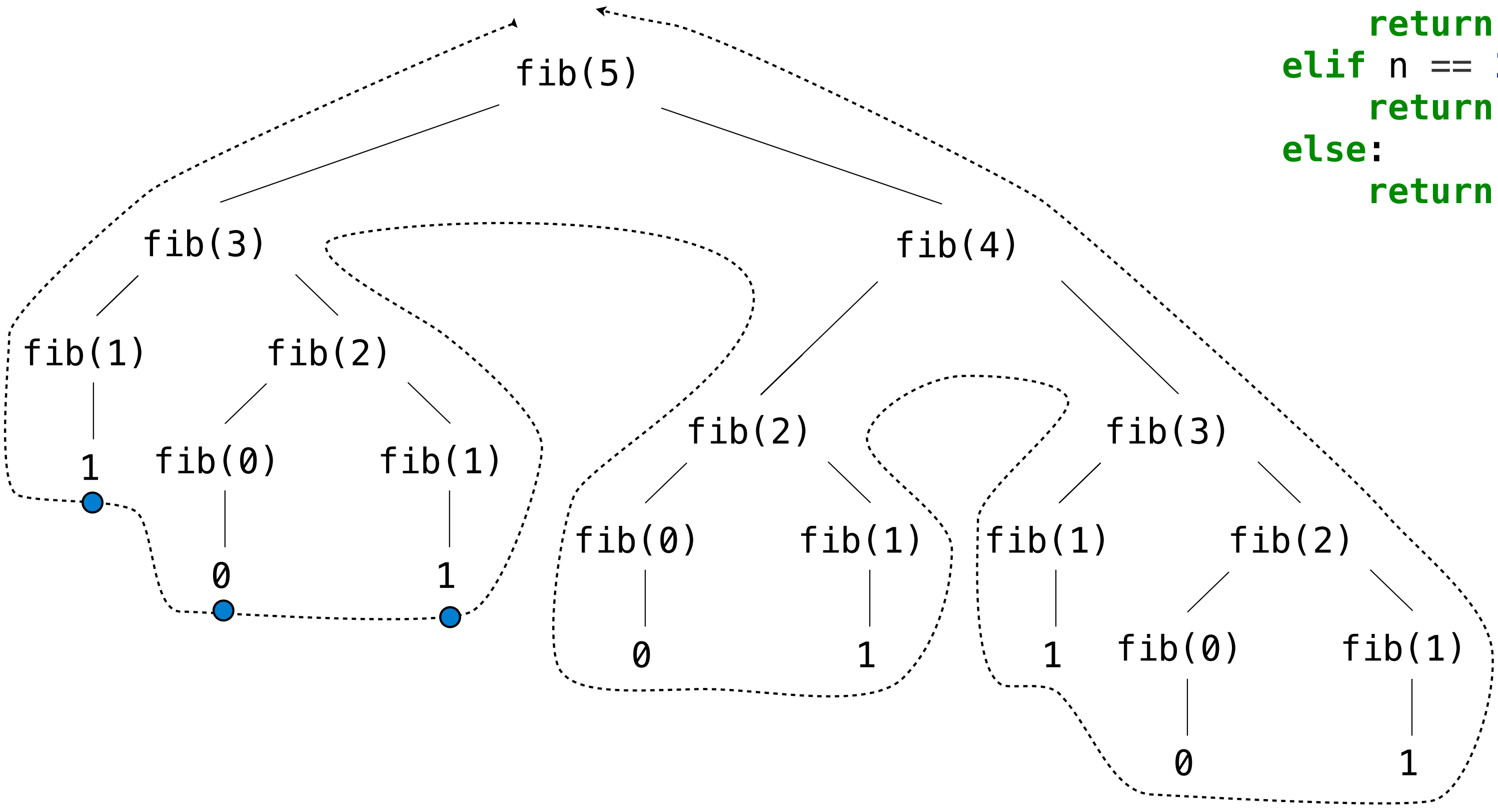

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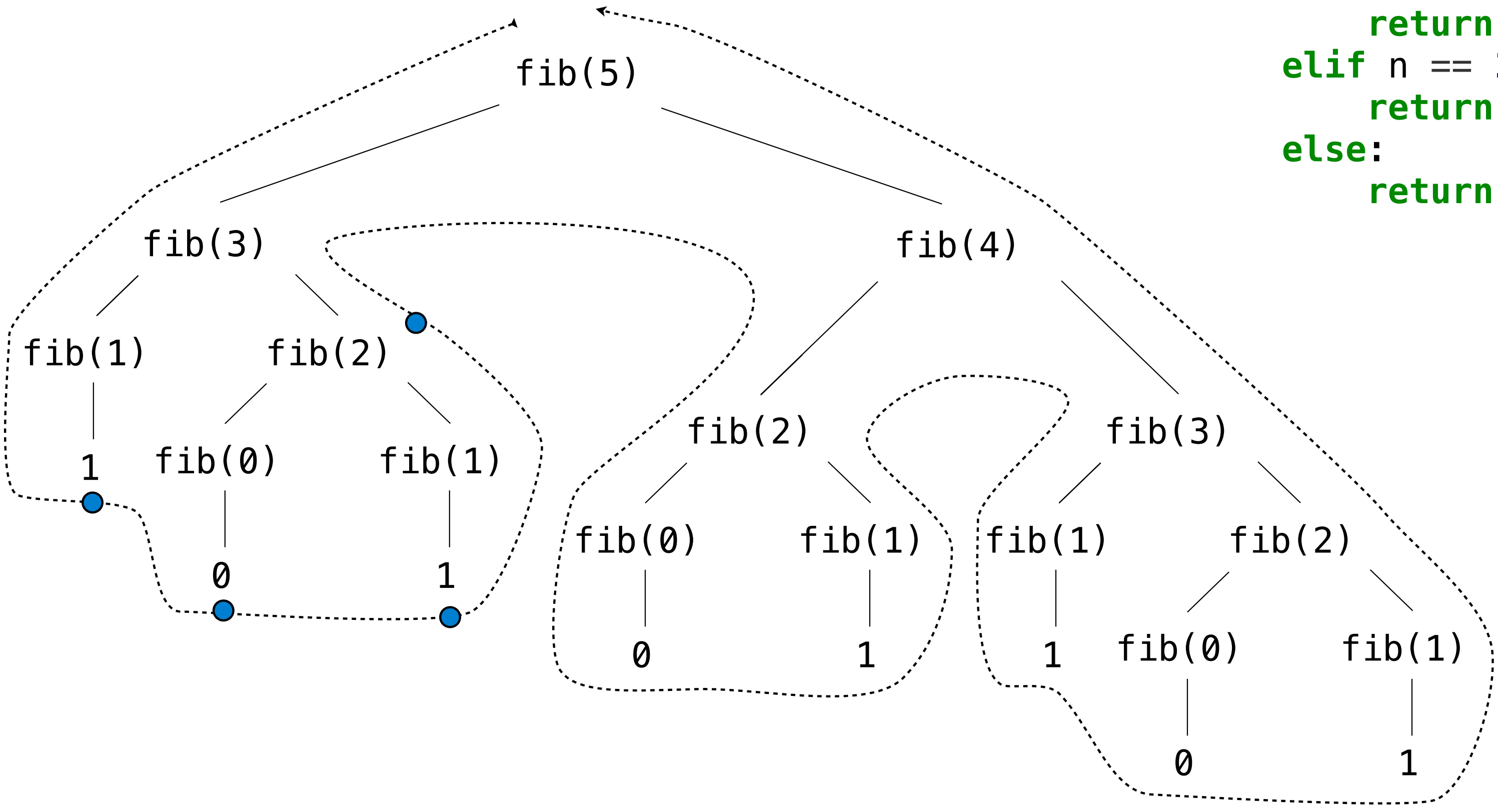

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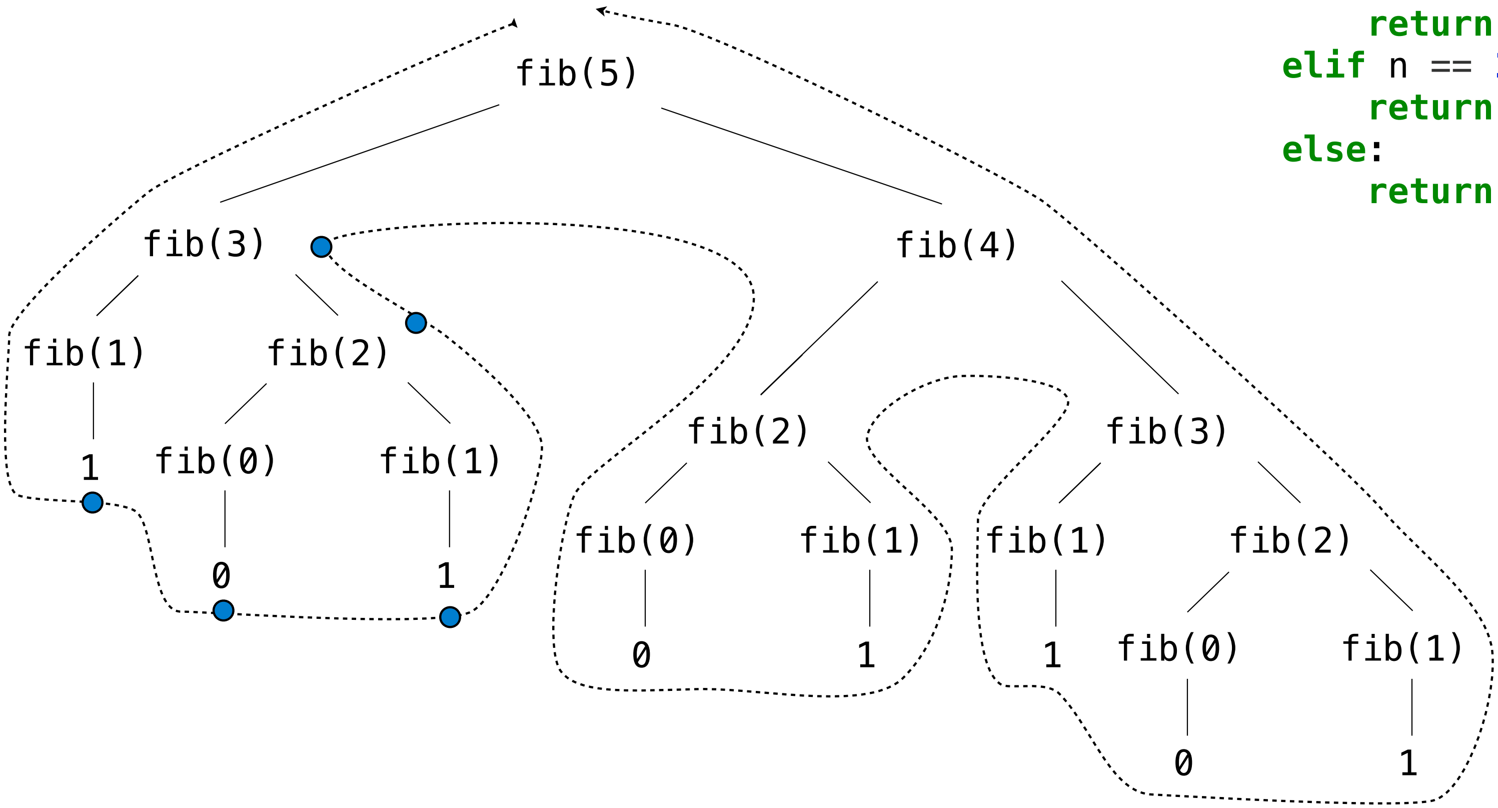

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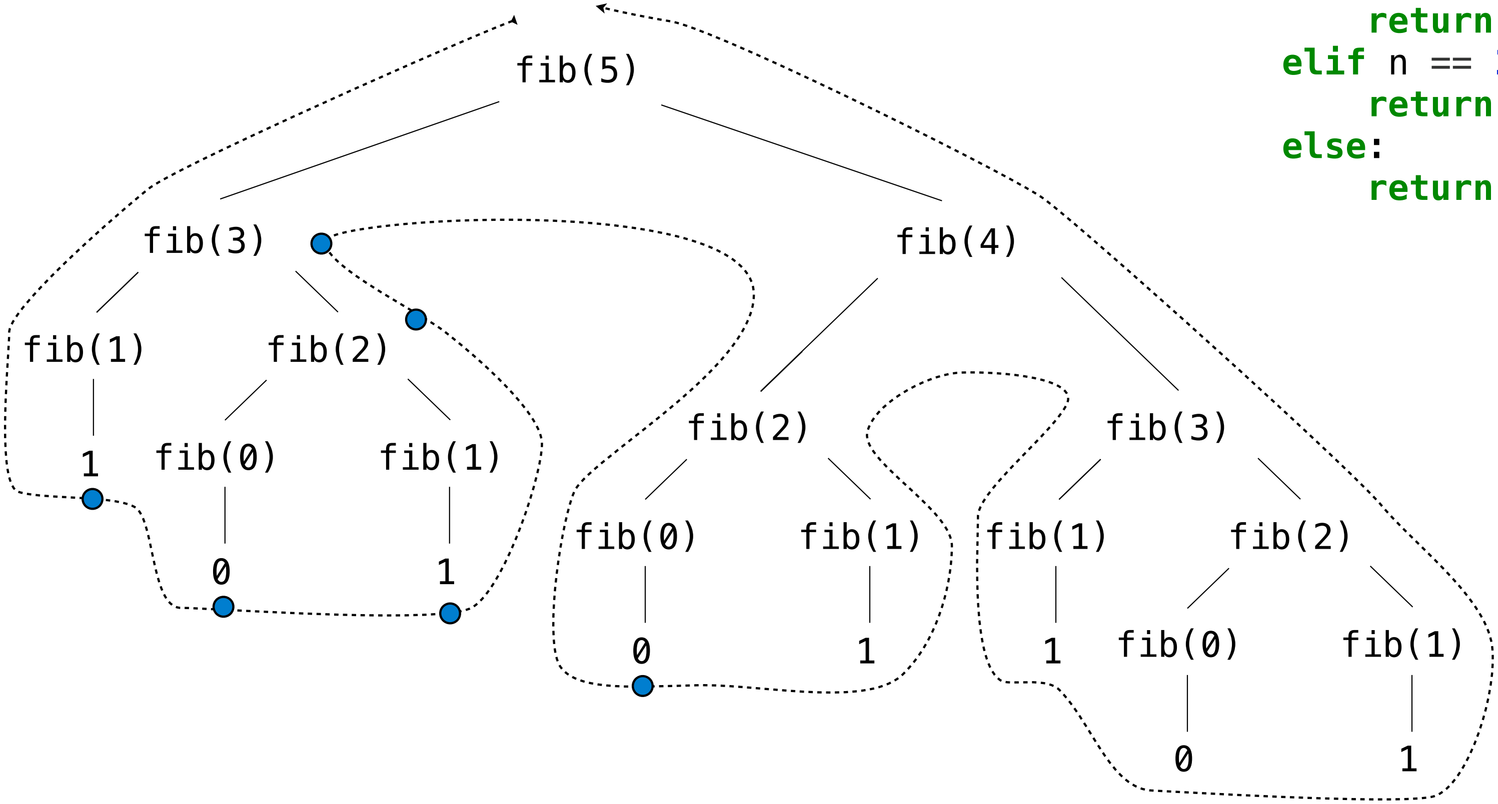


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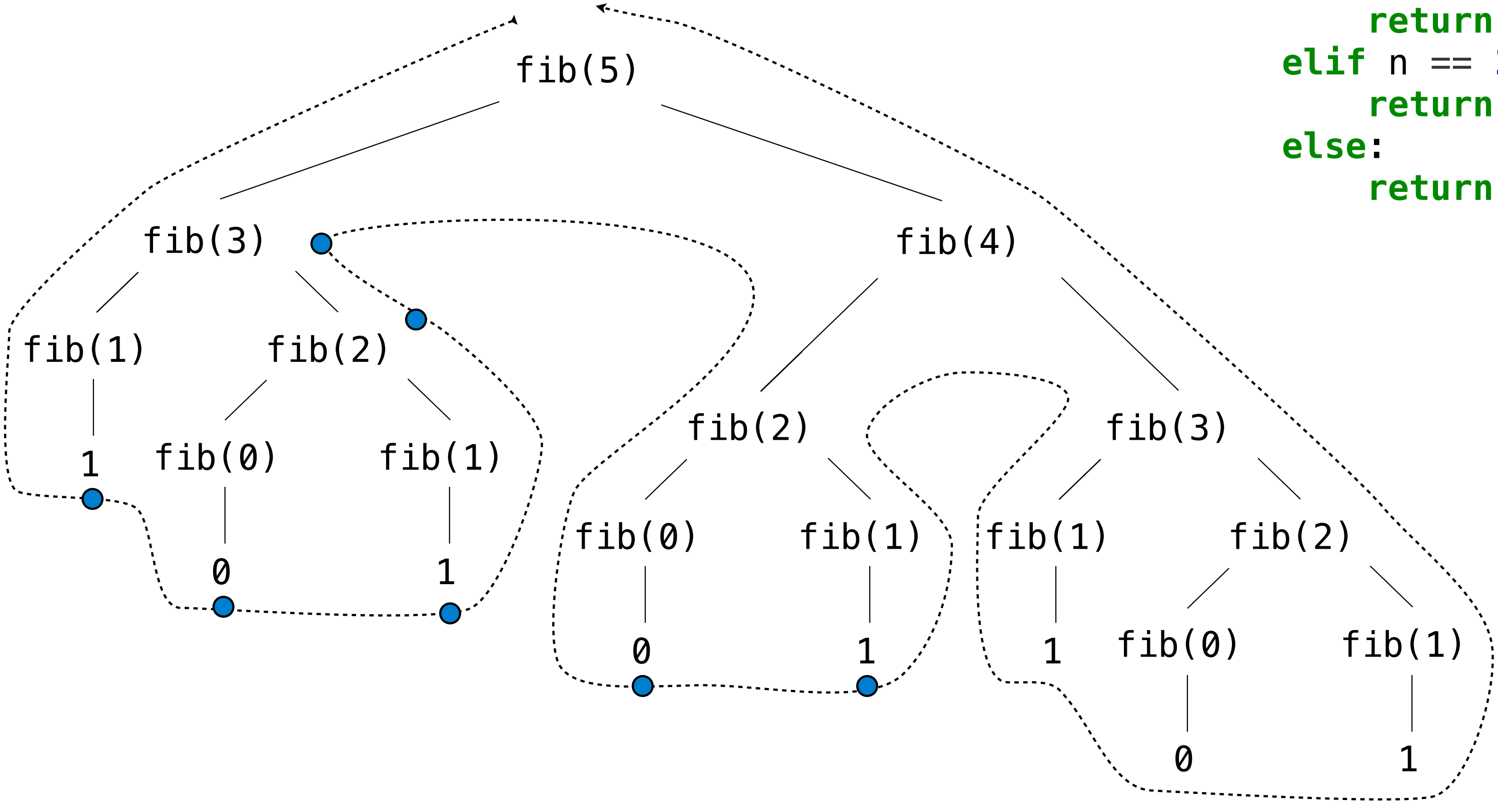

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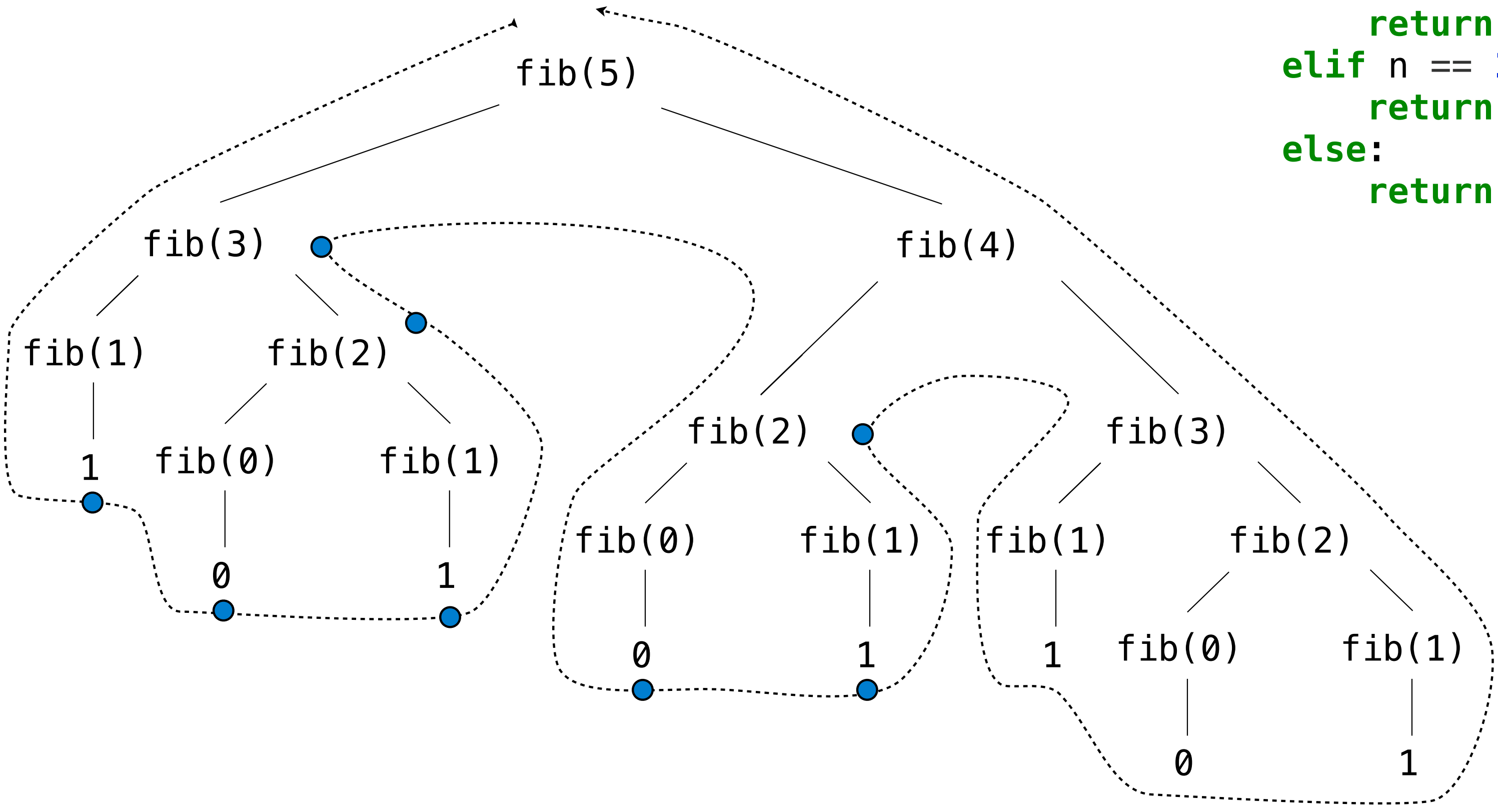

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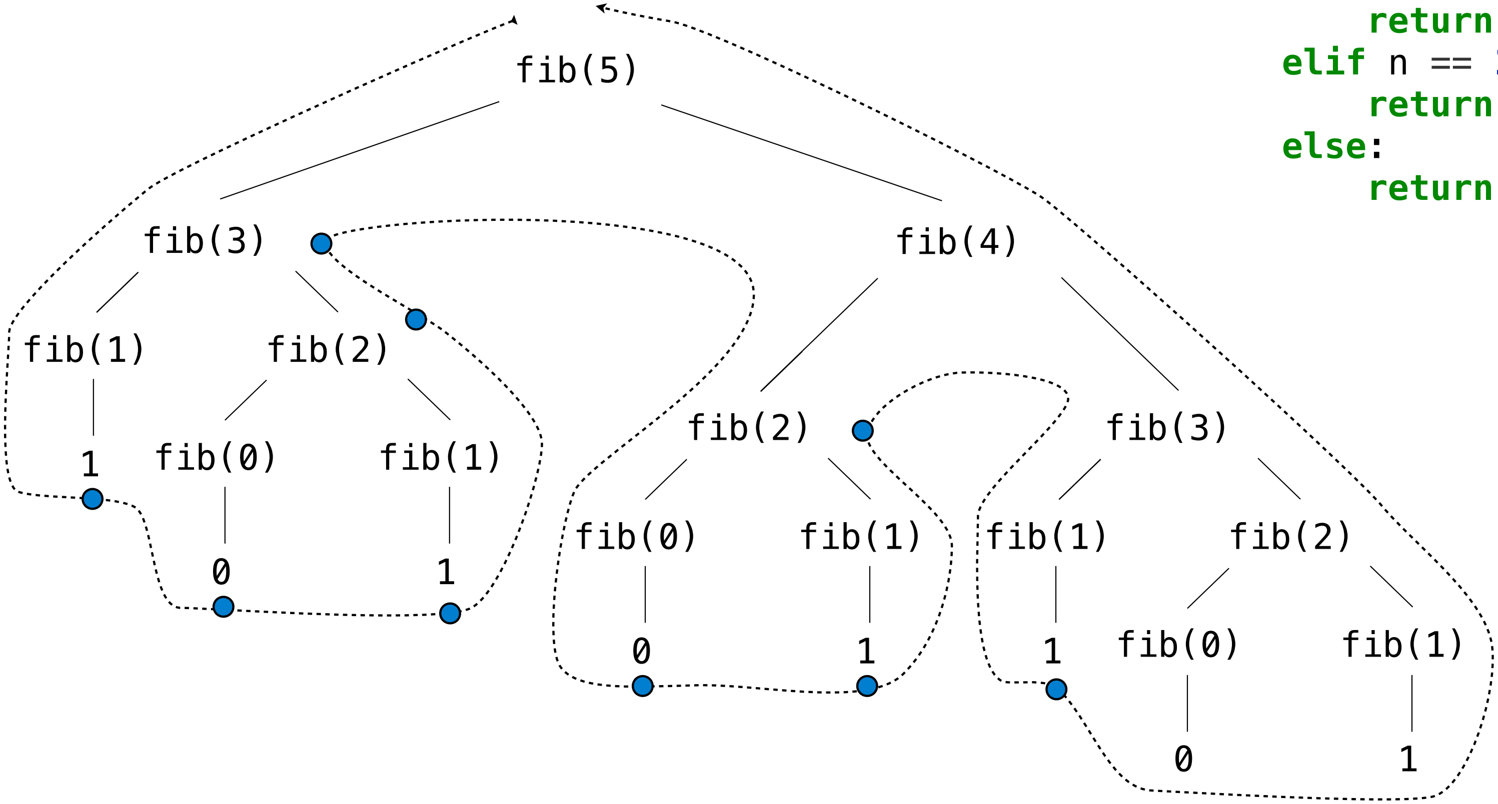

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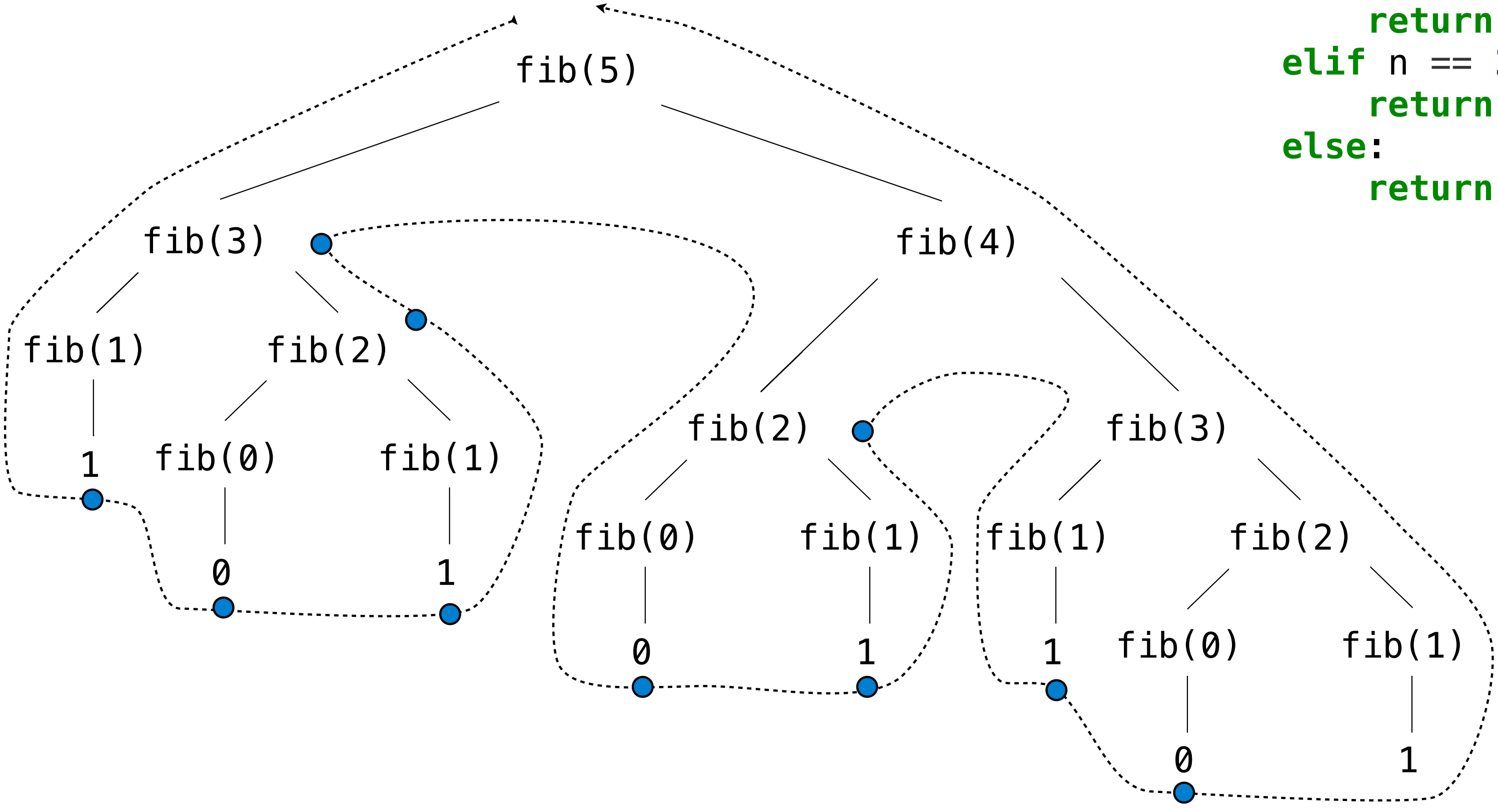

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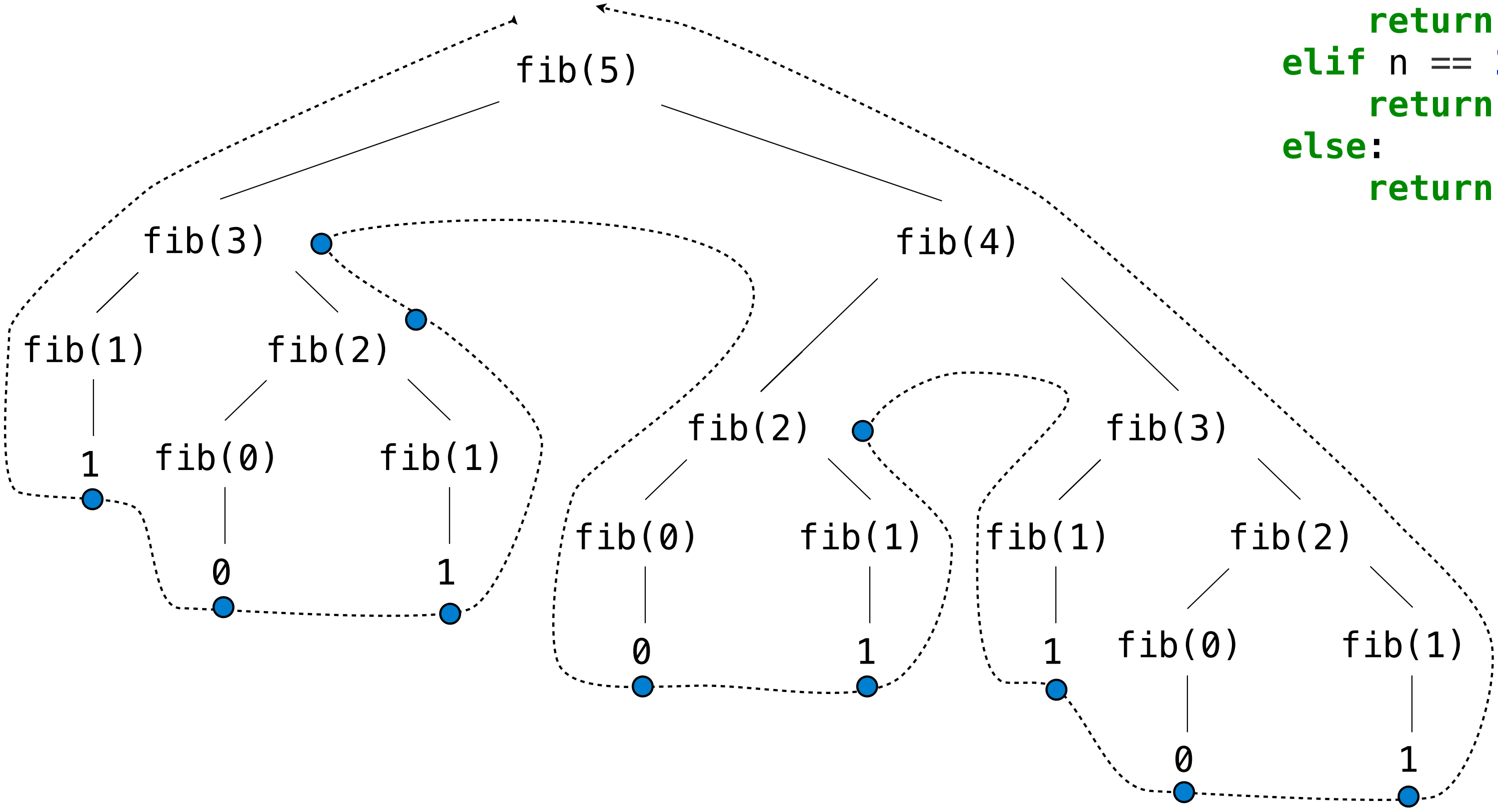

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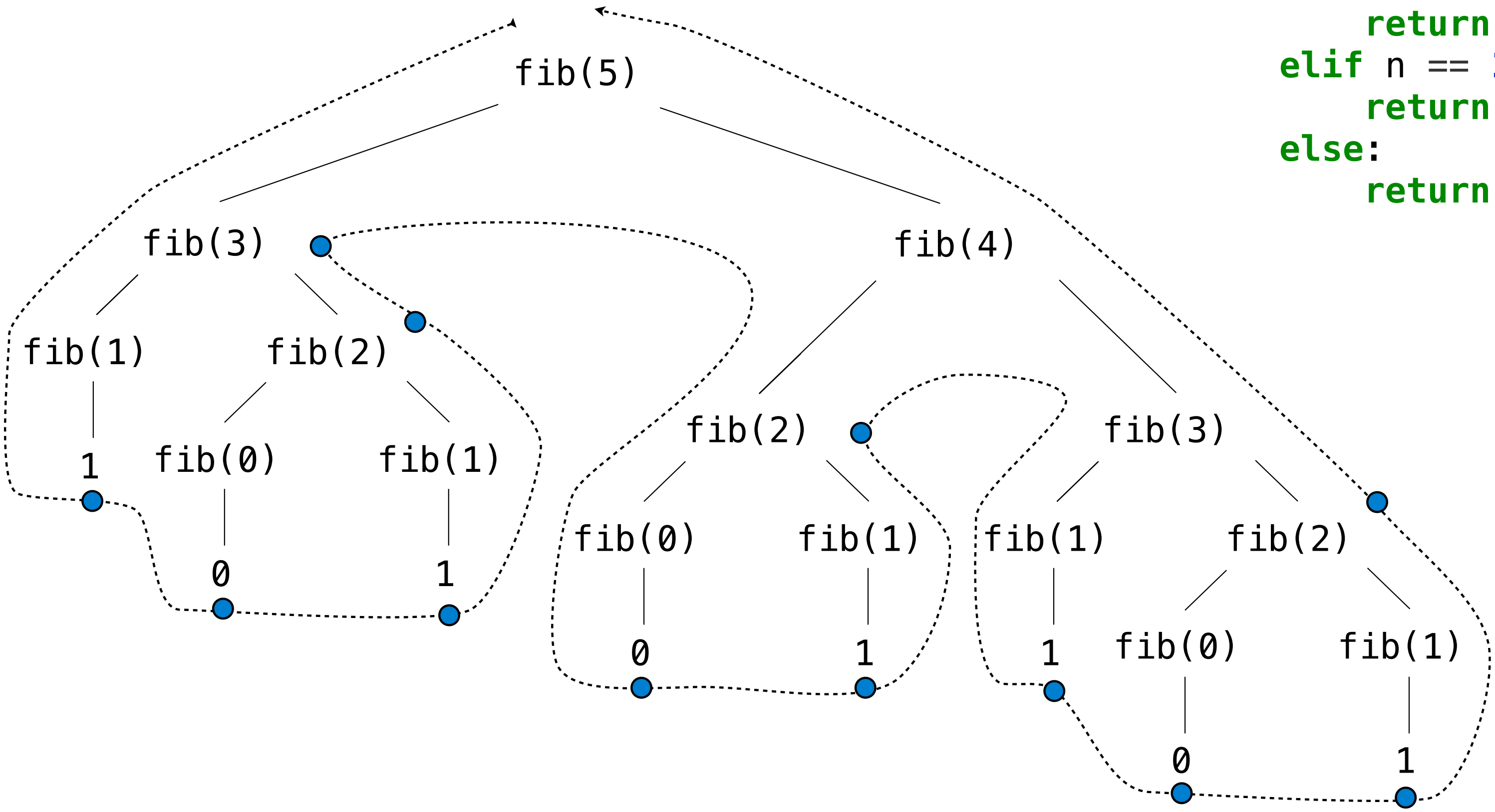

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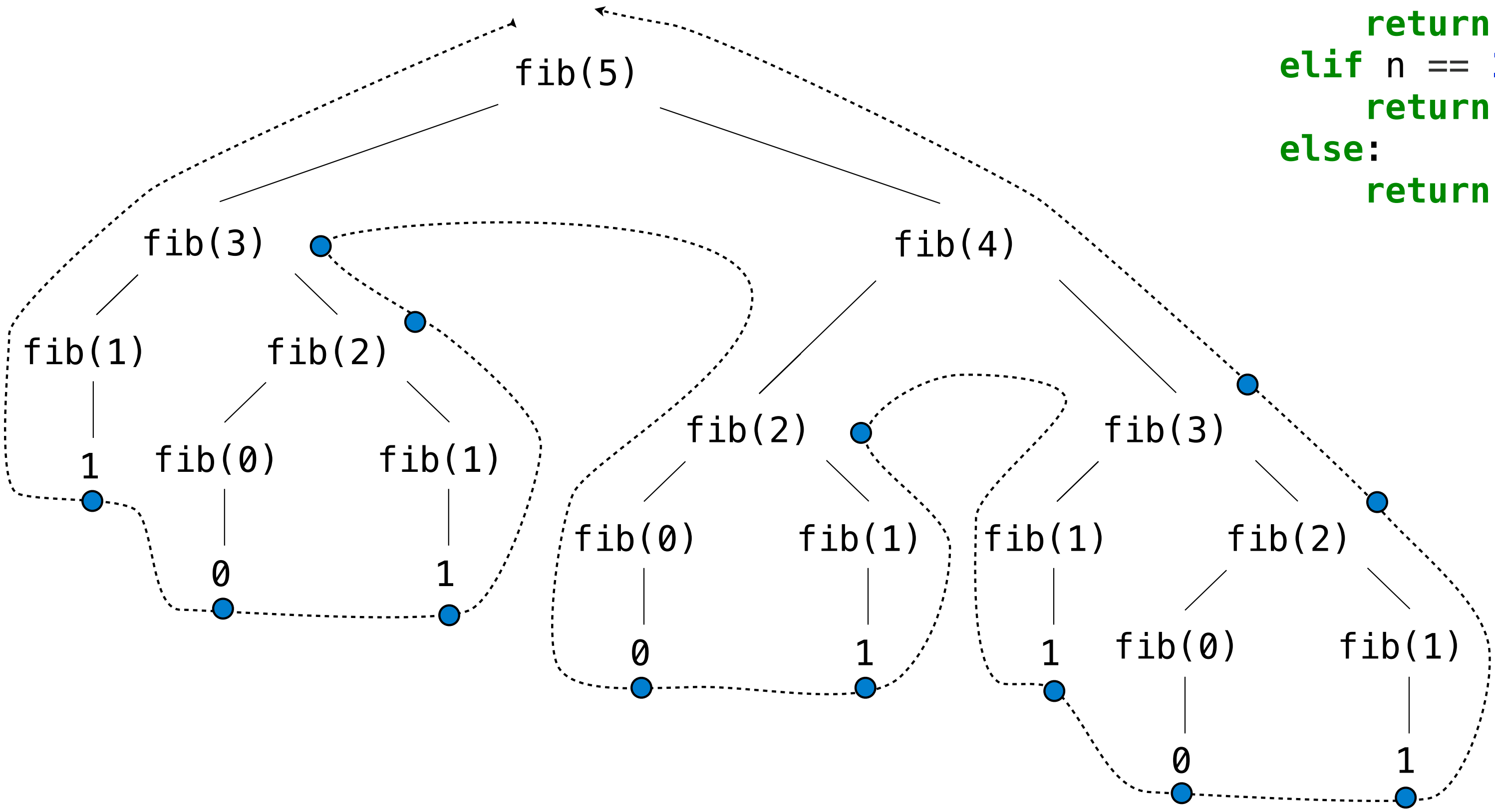

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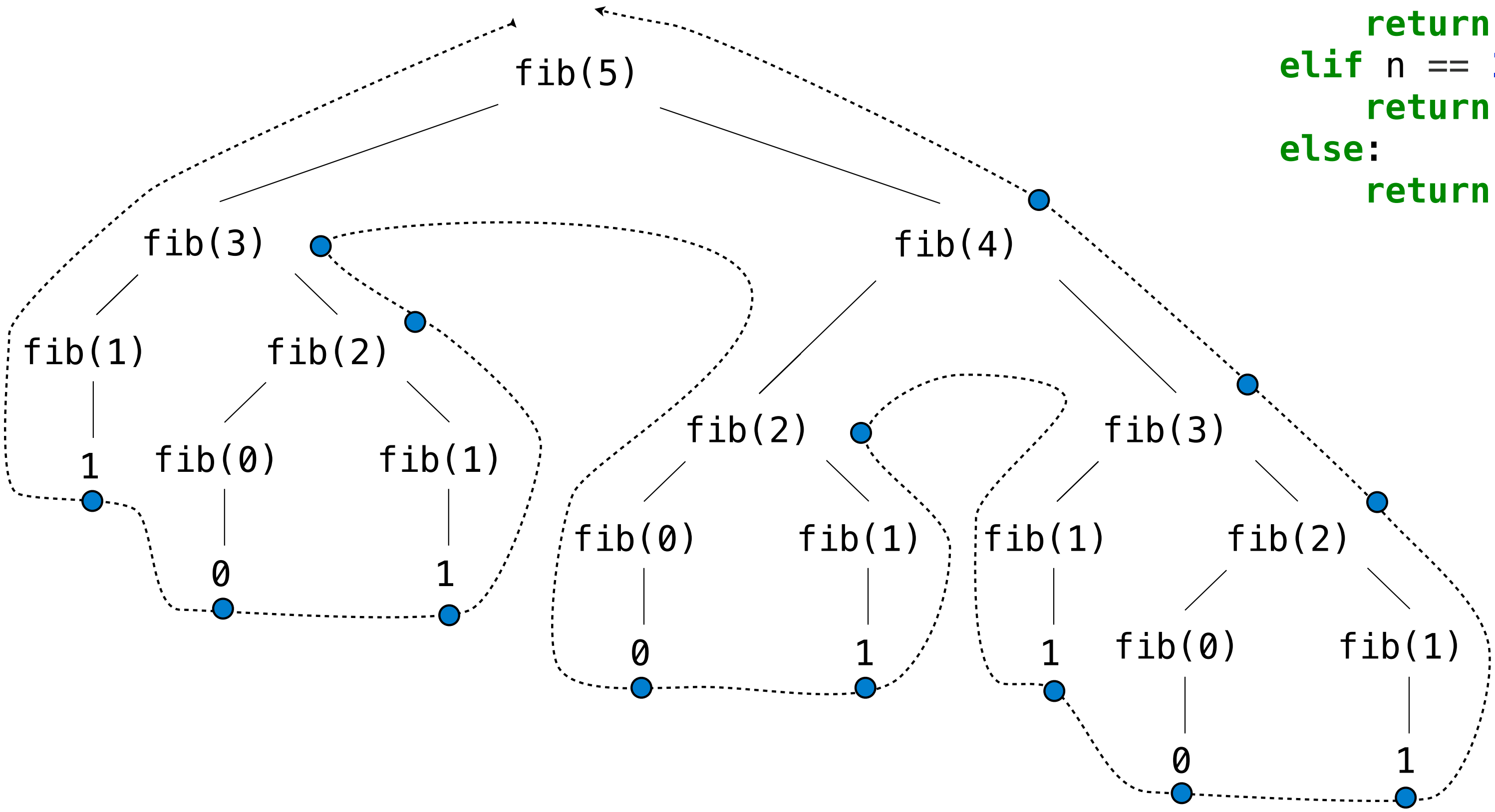


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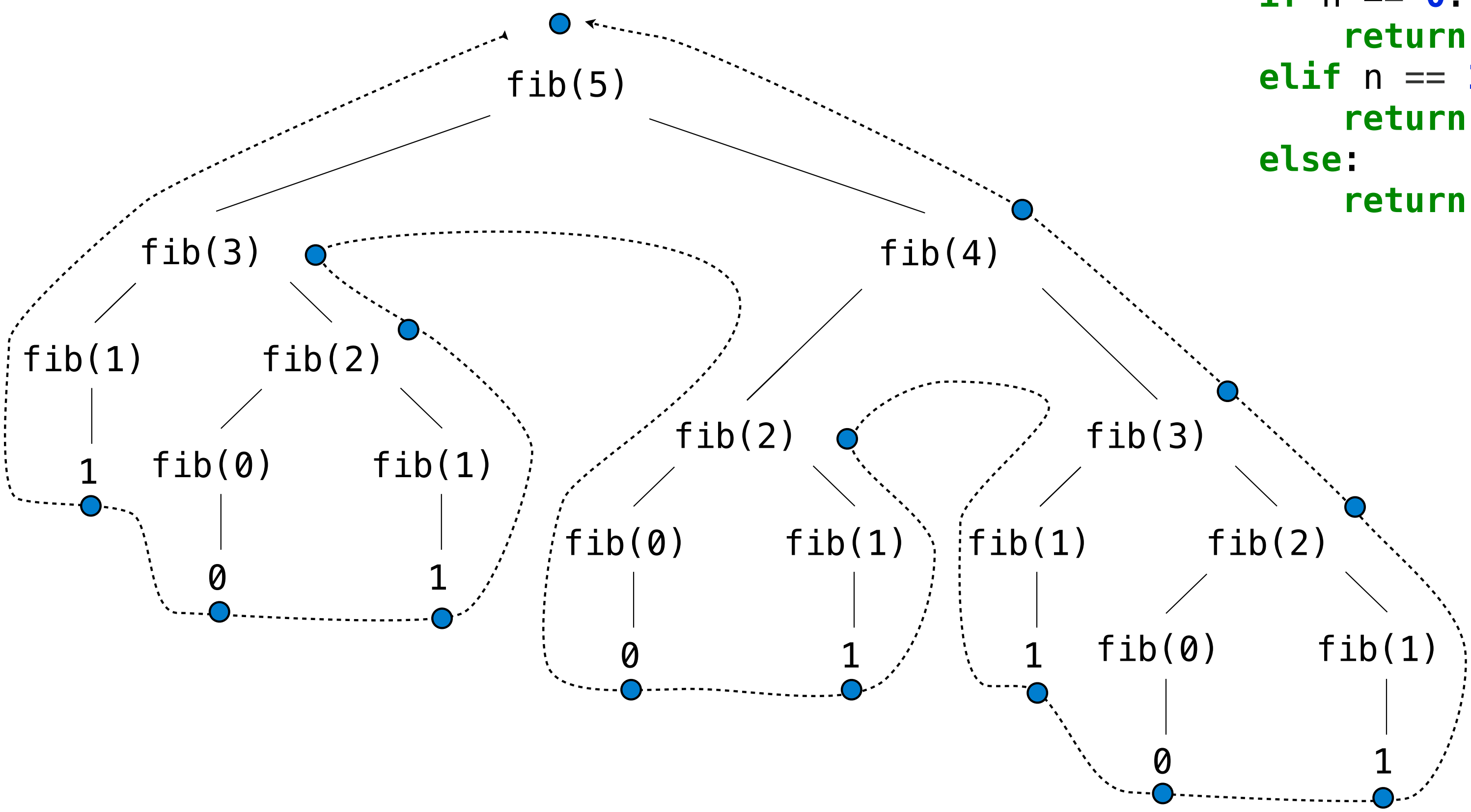

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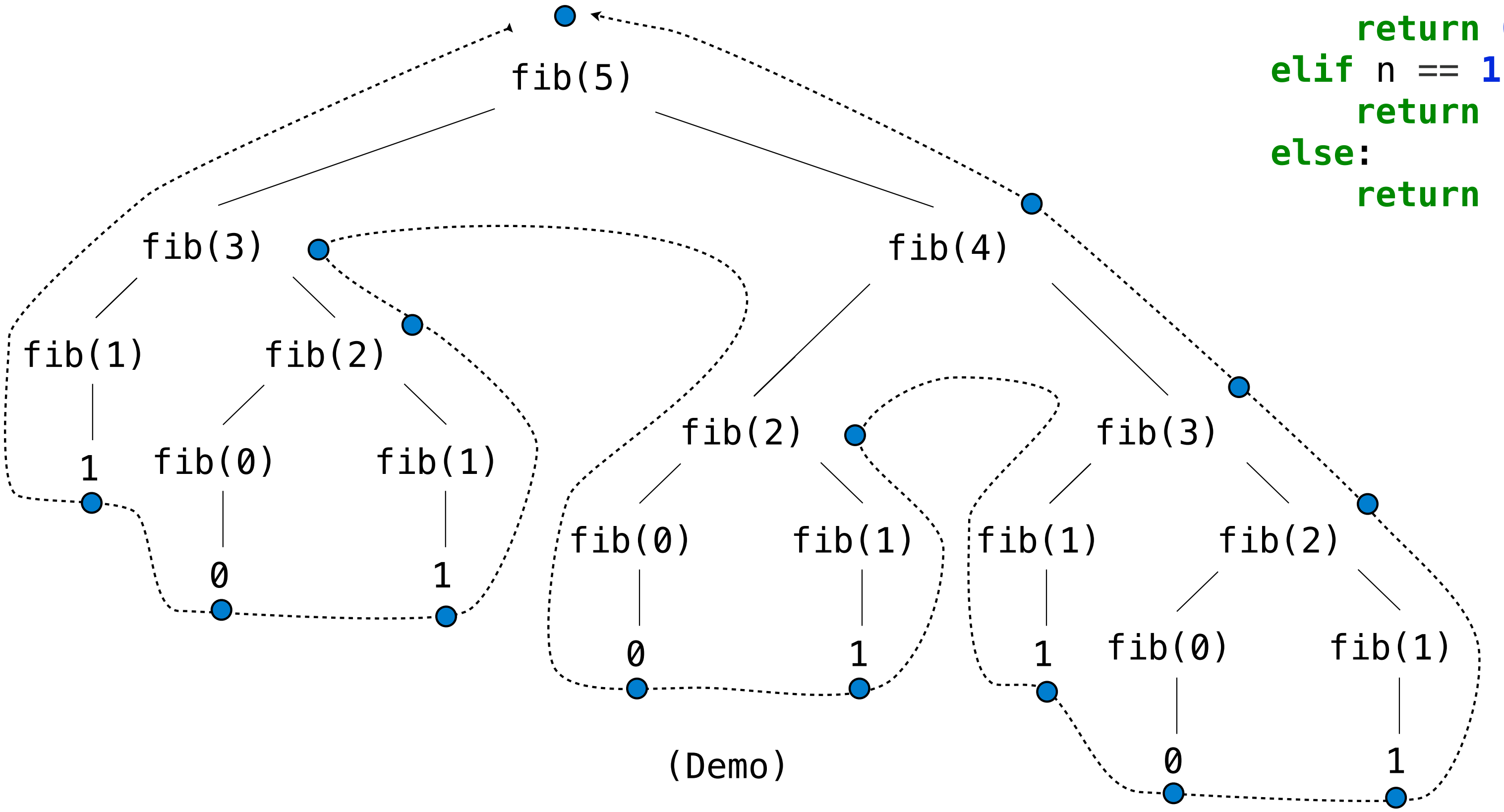

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(Demo)

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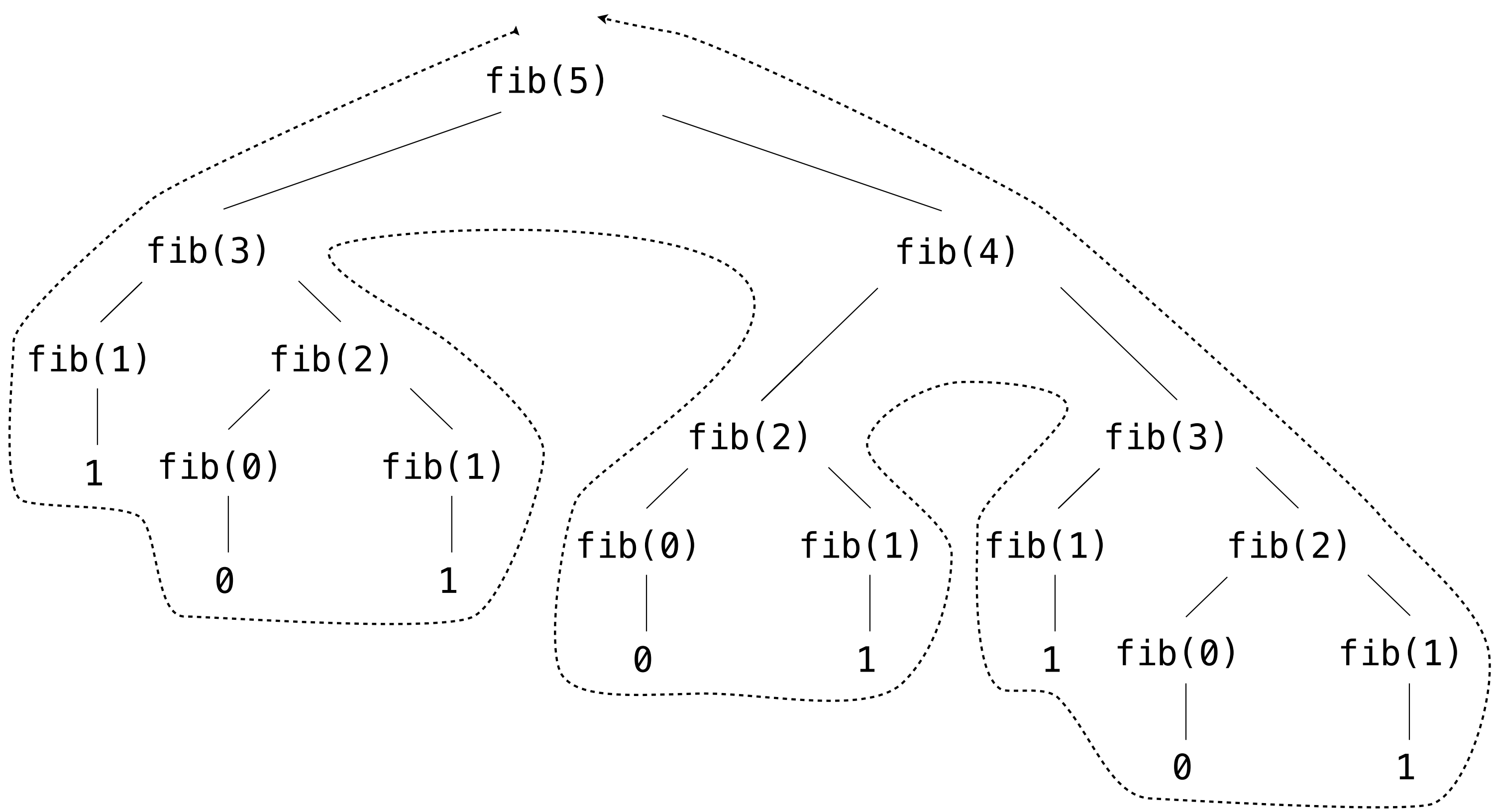

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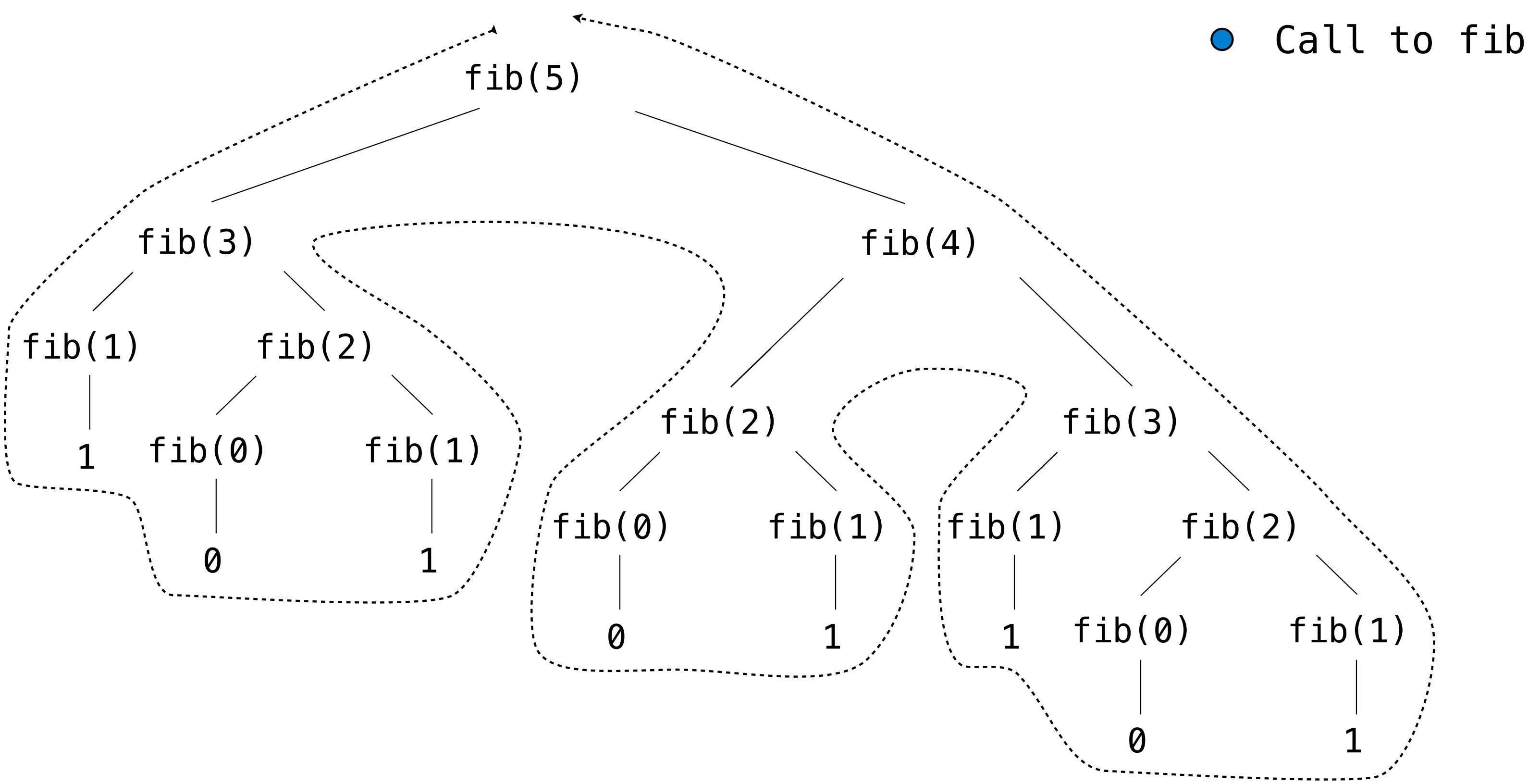

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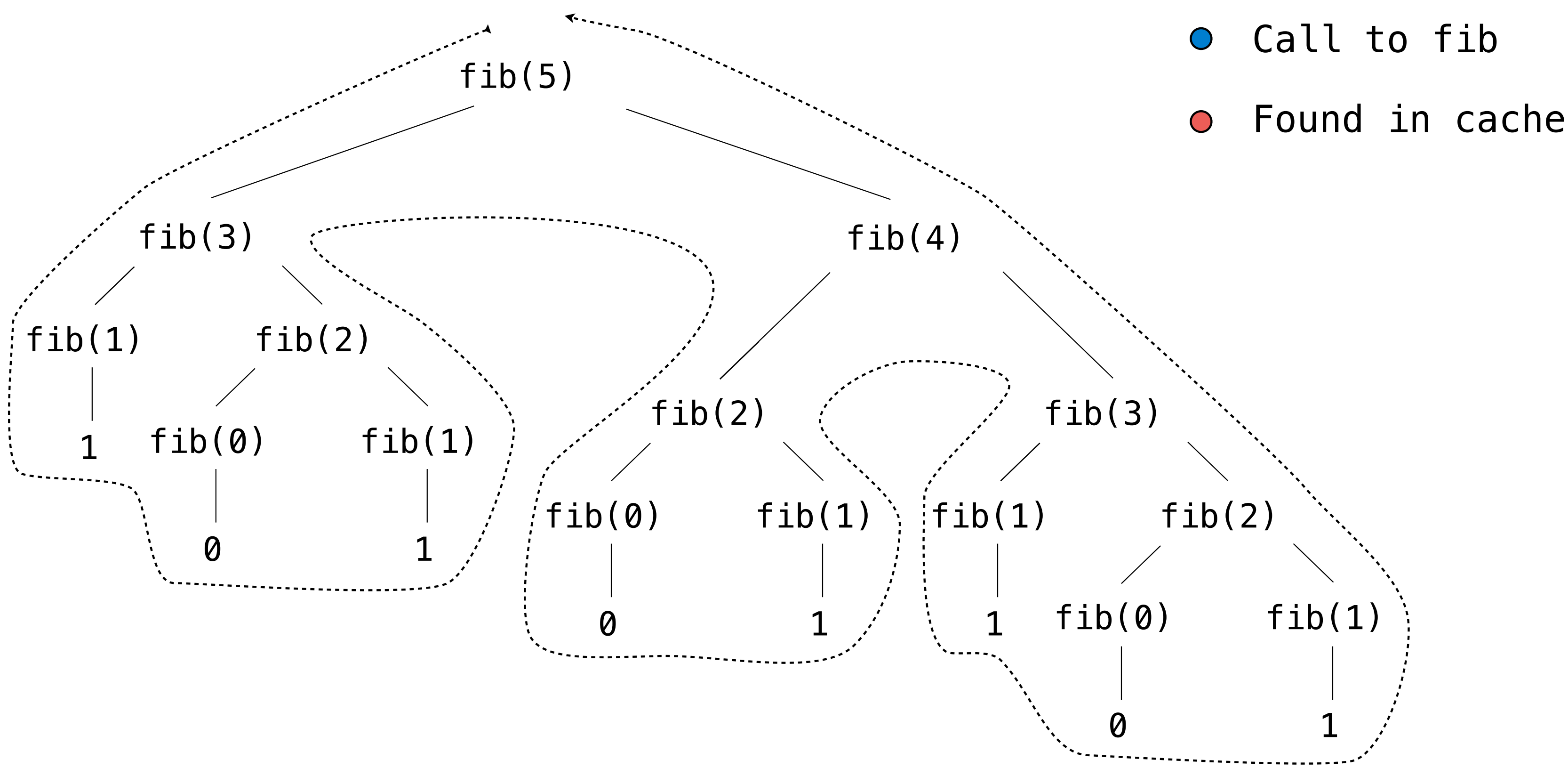

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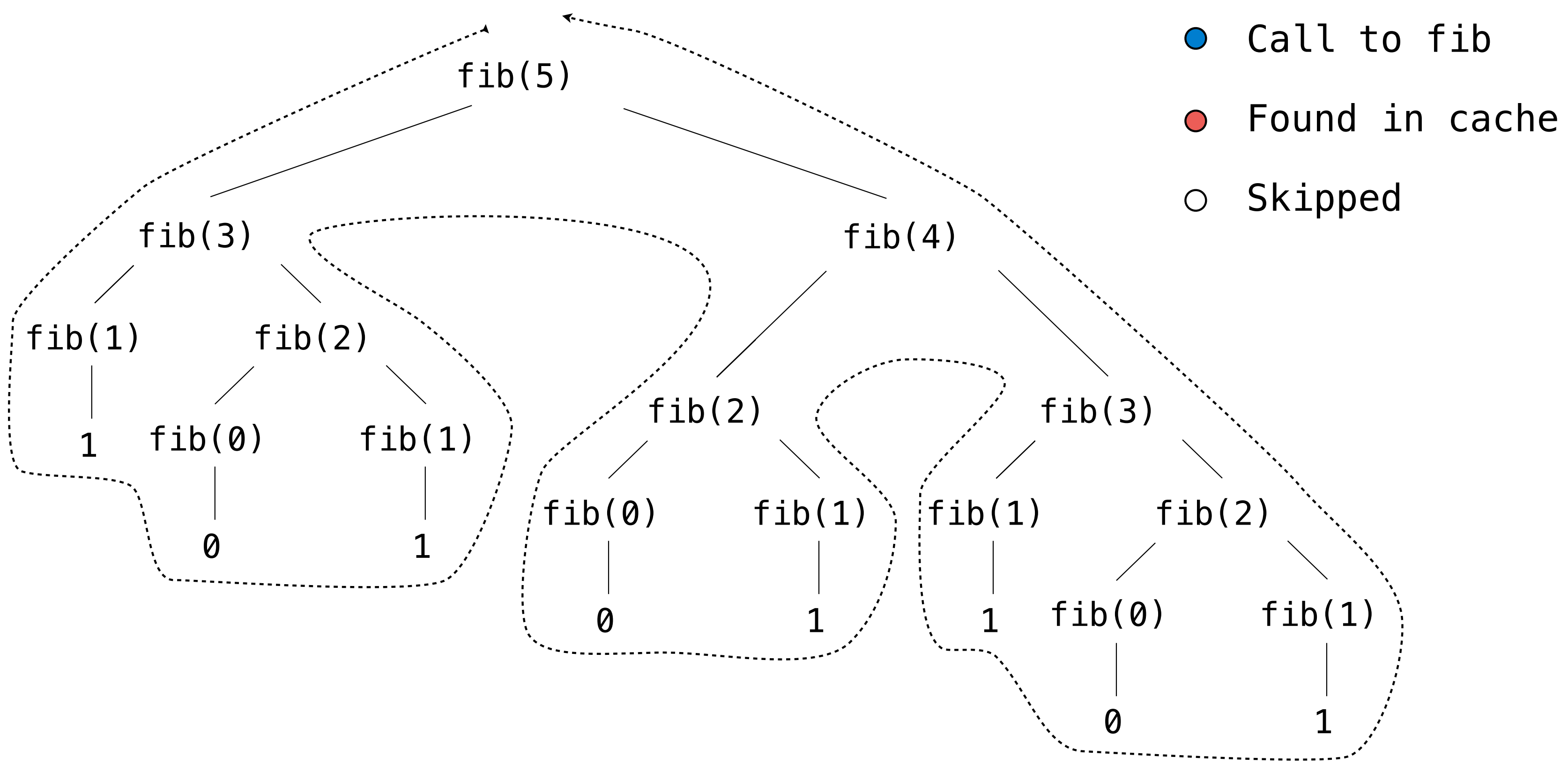

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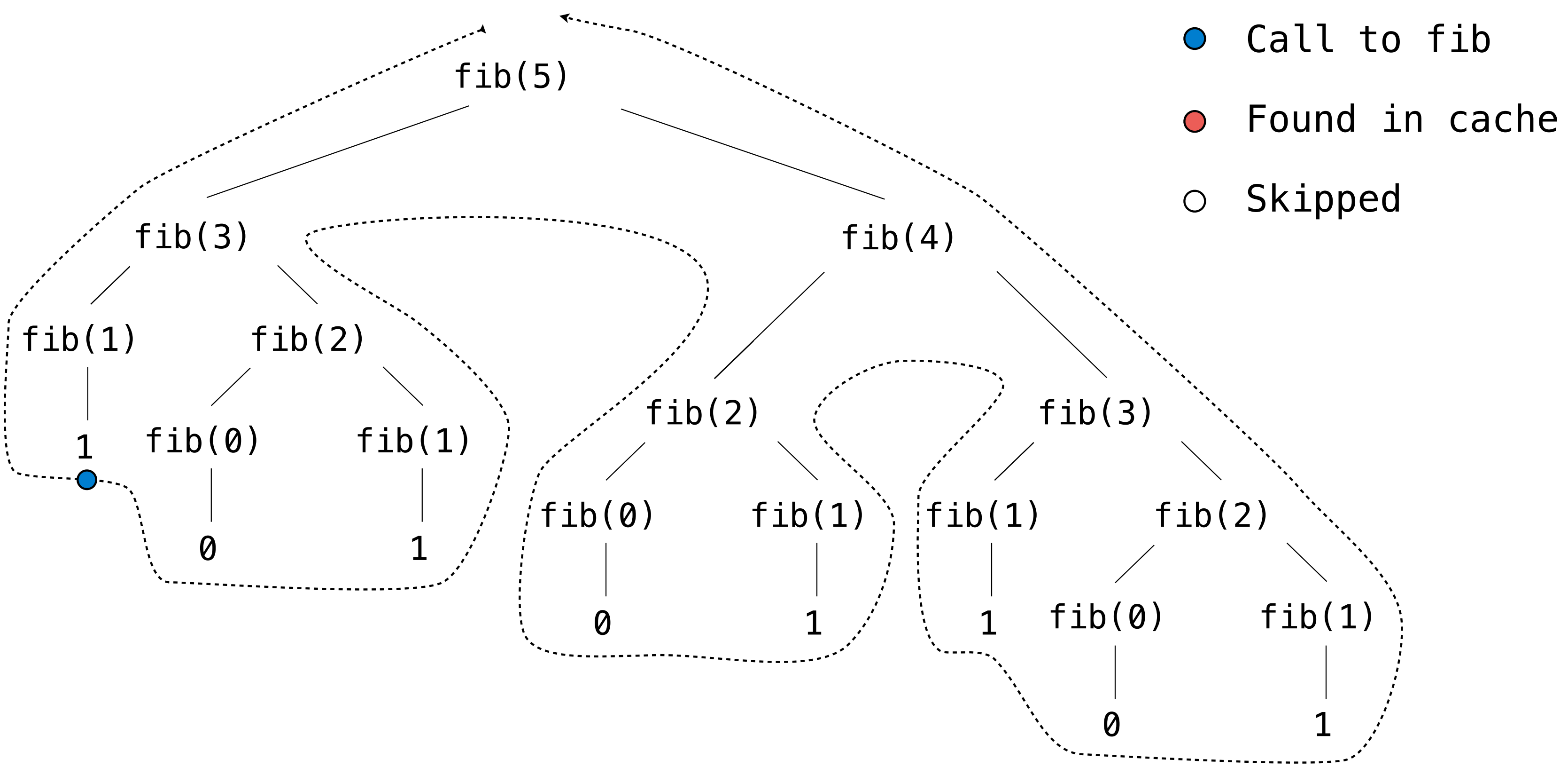

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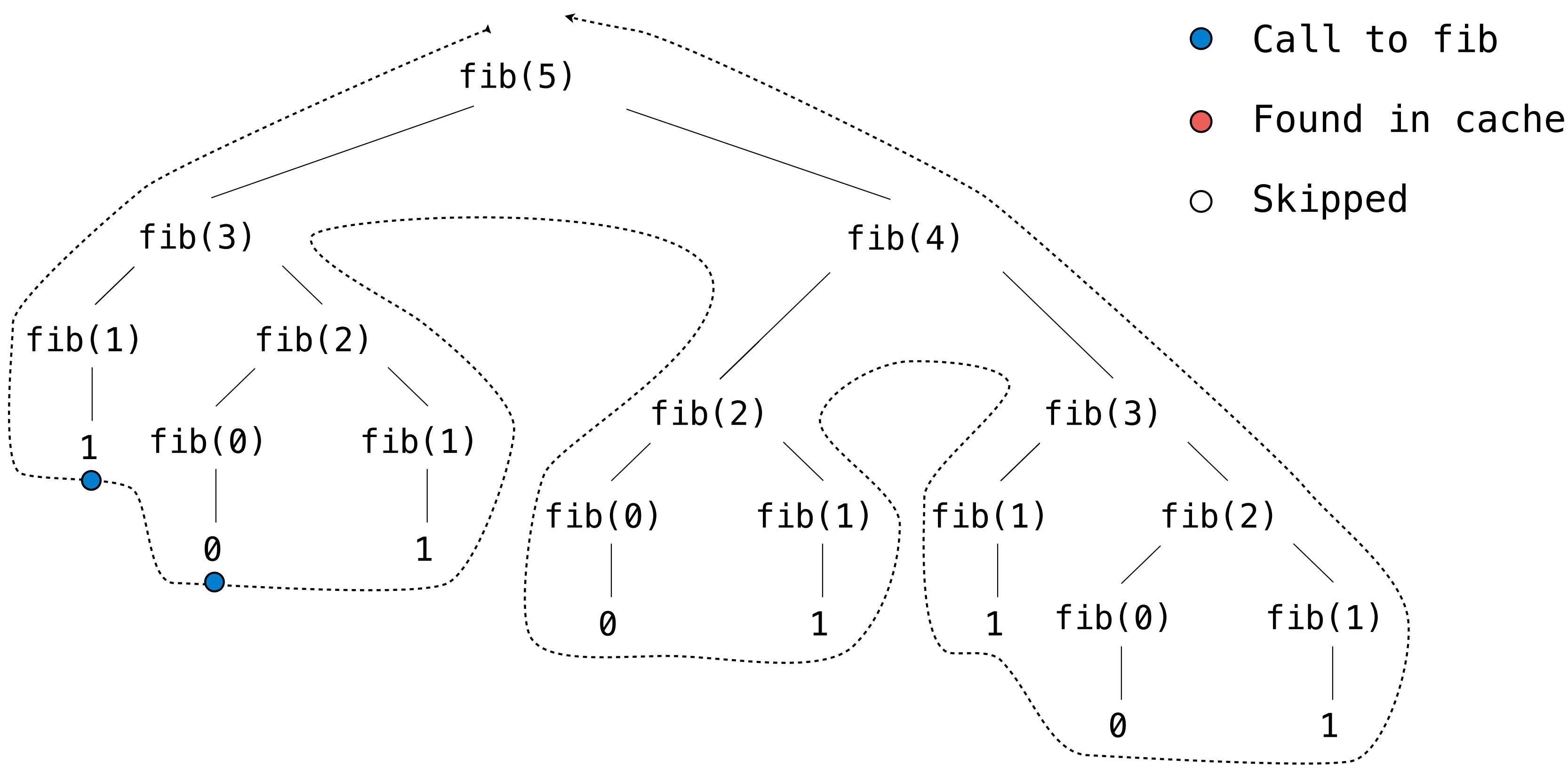

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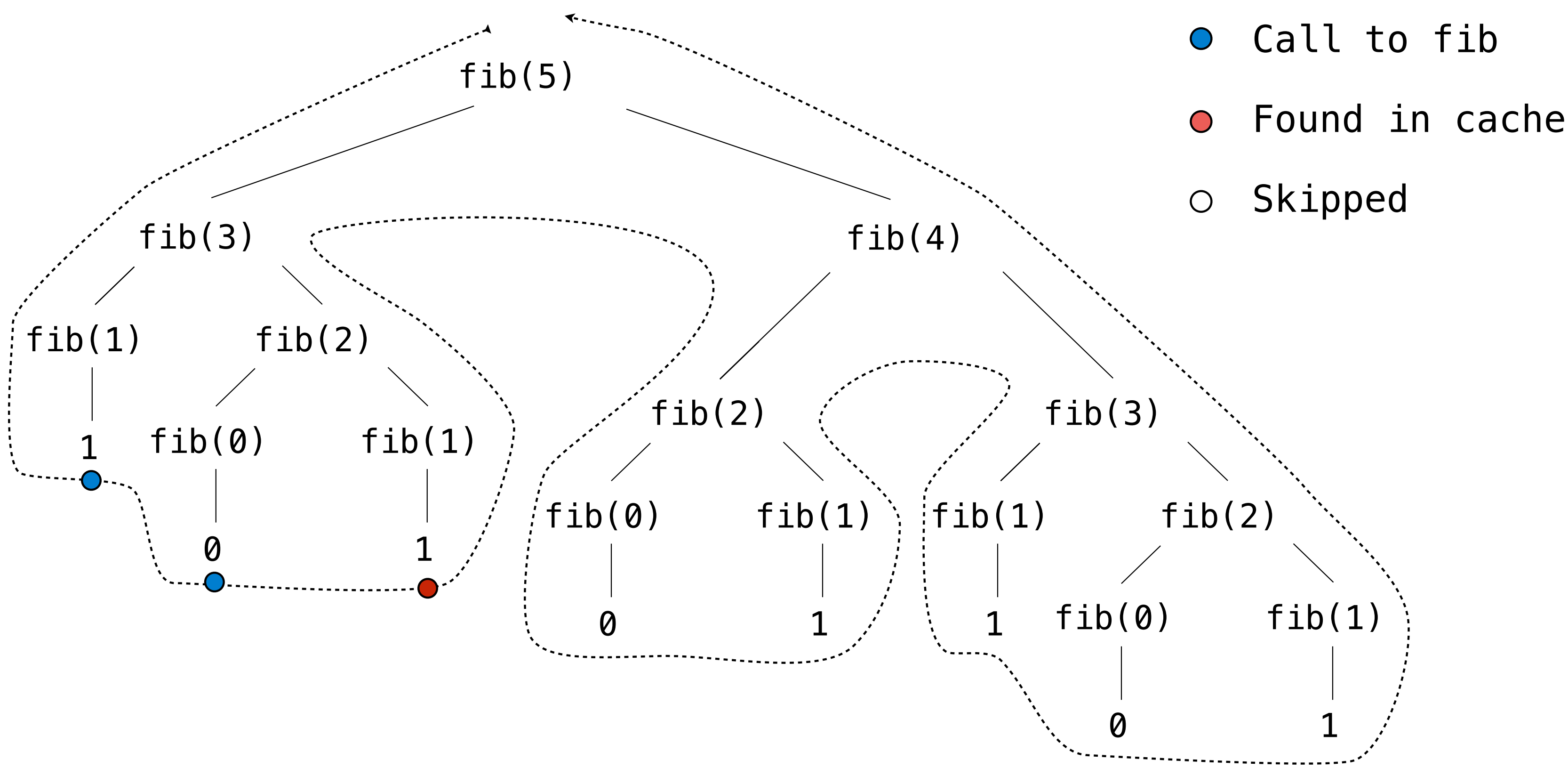

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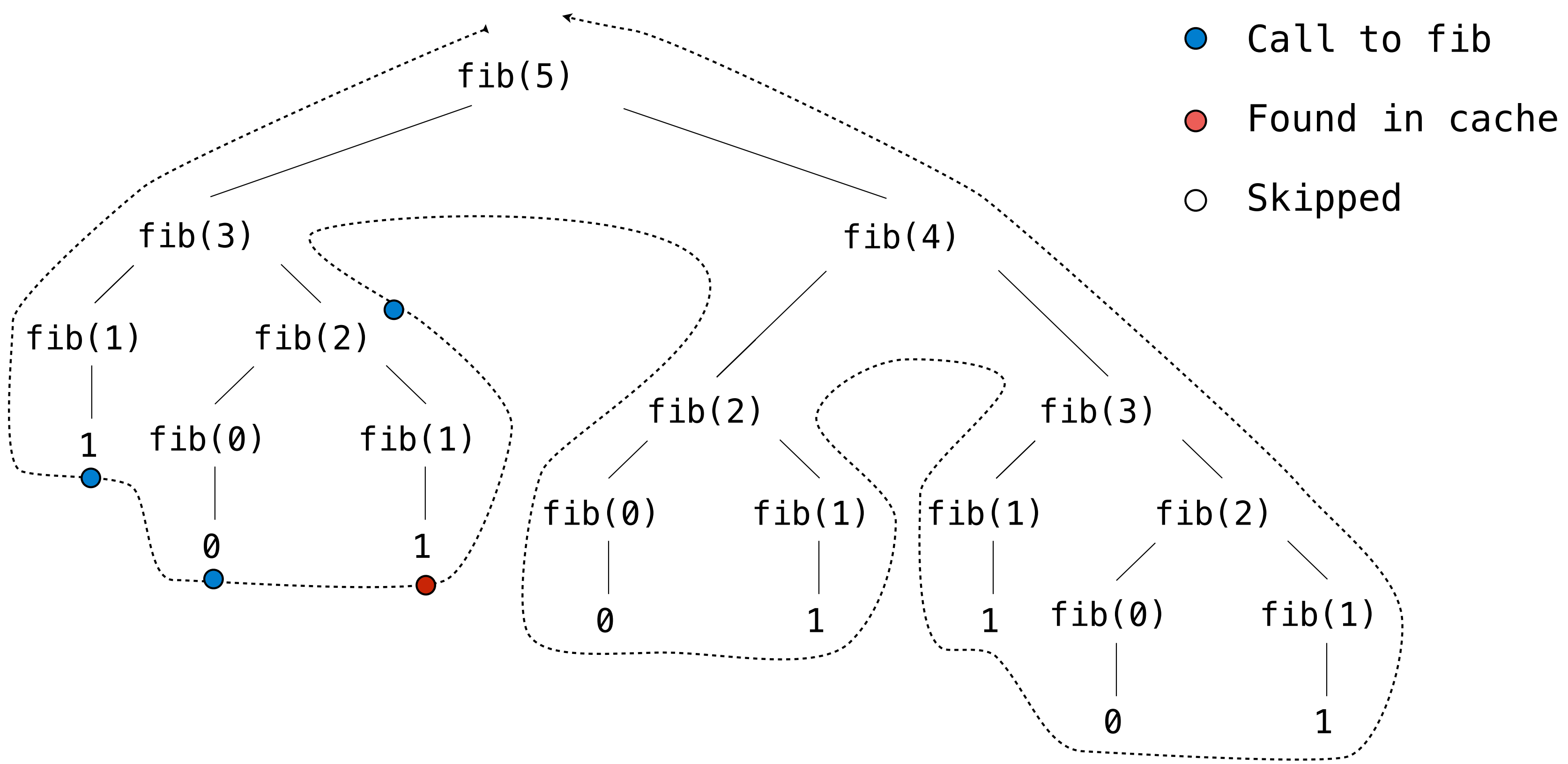

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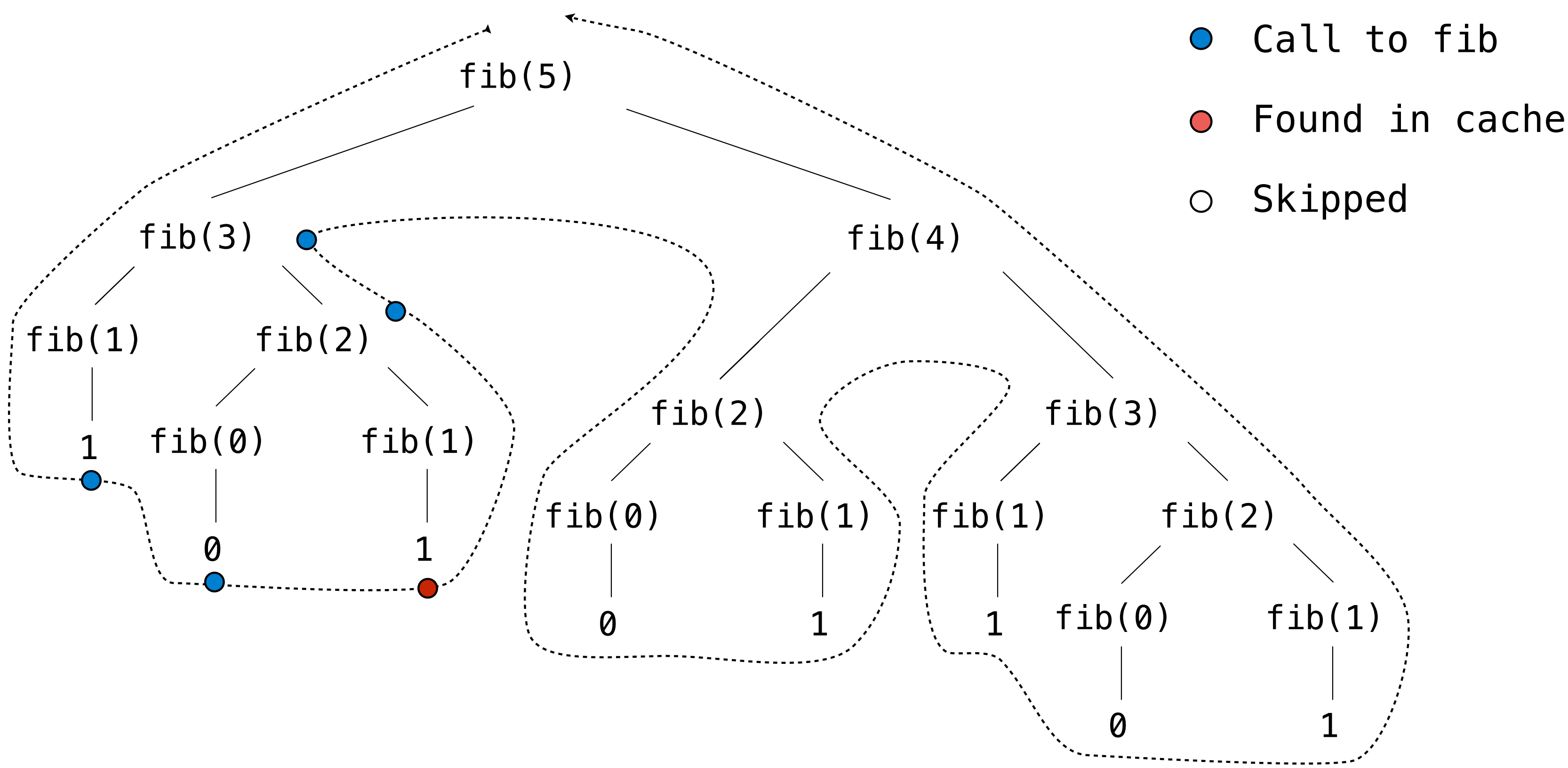

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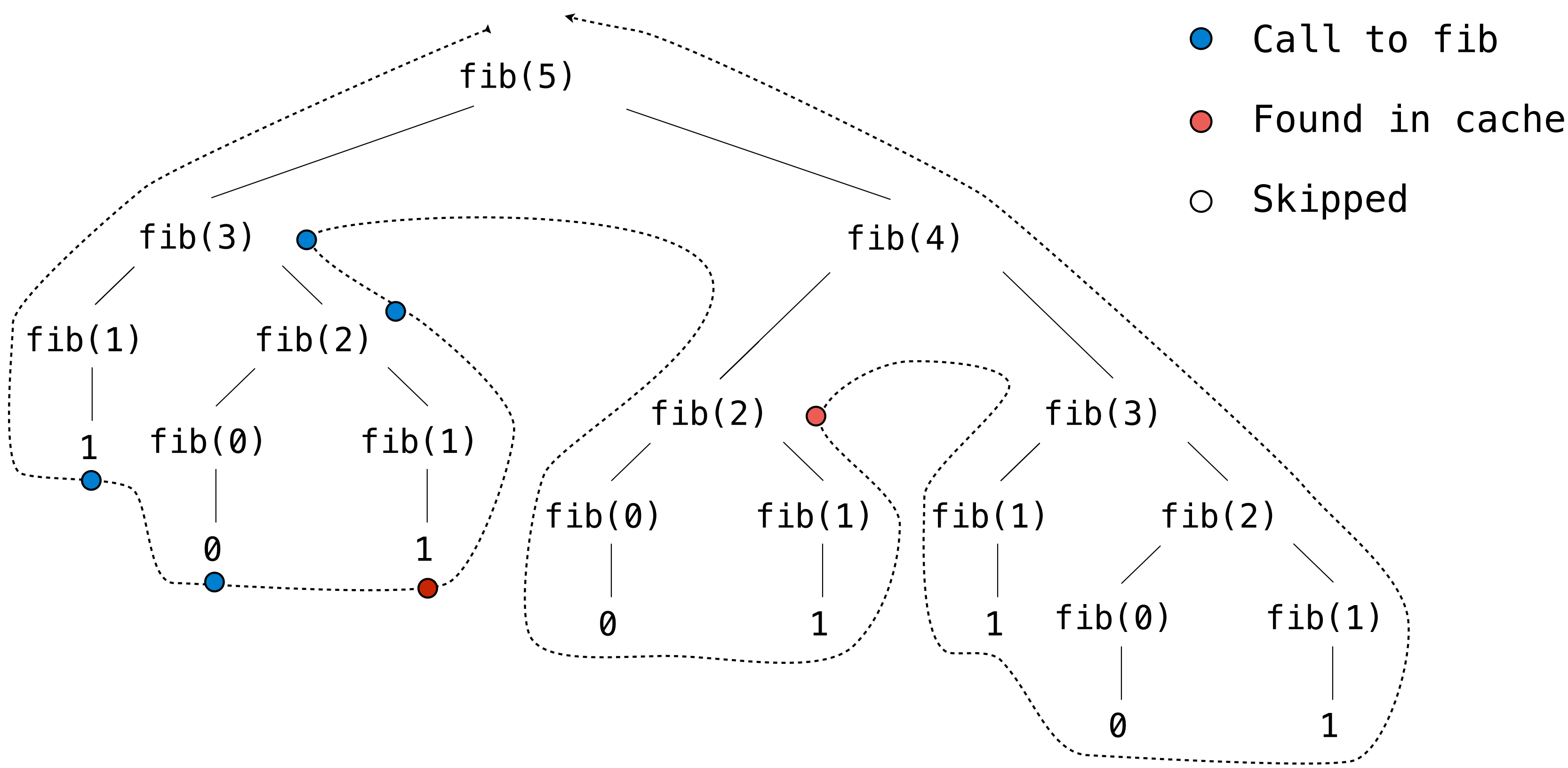

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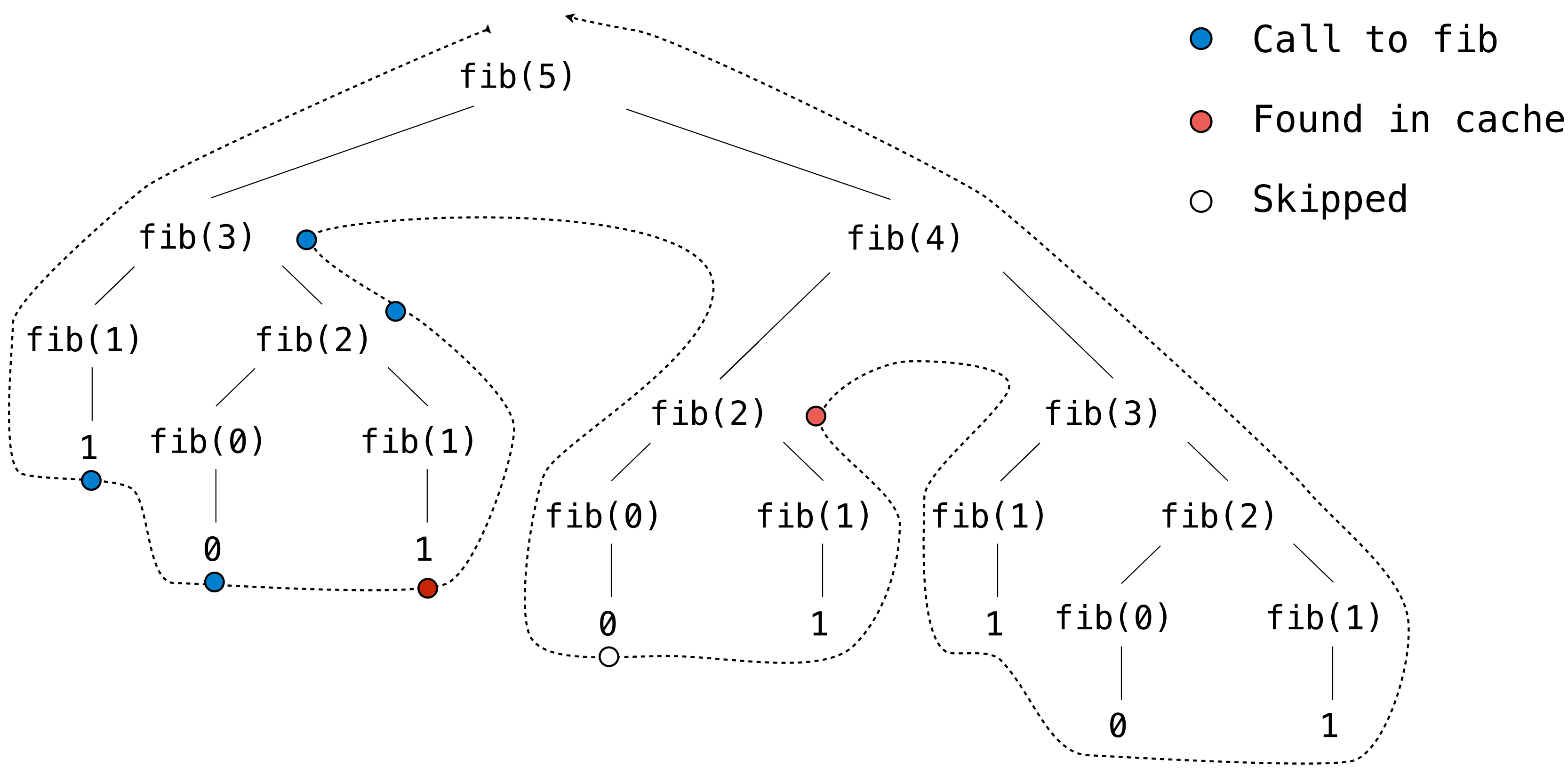

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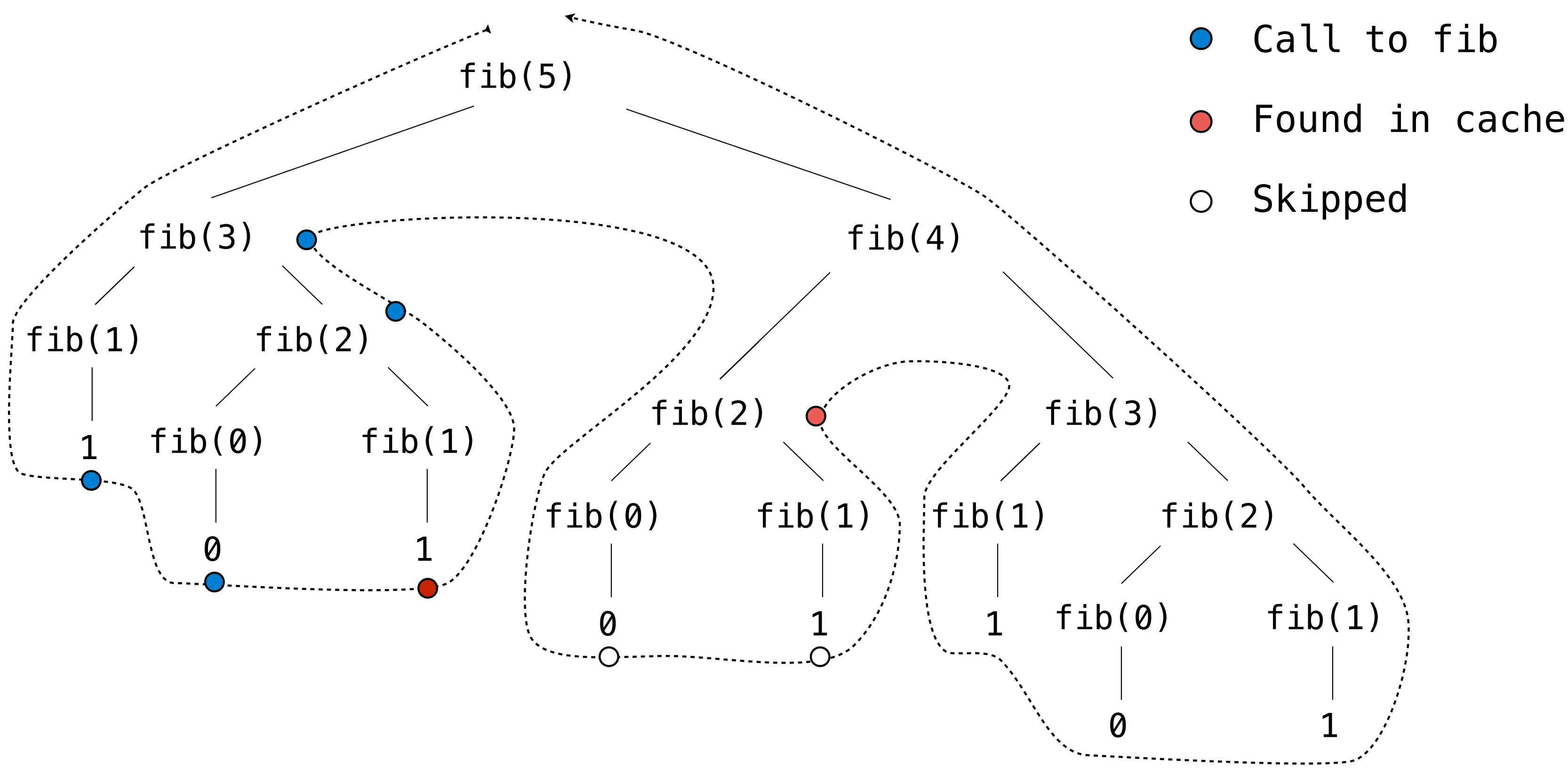

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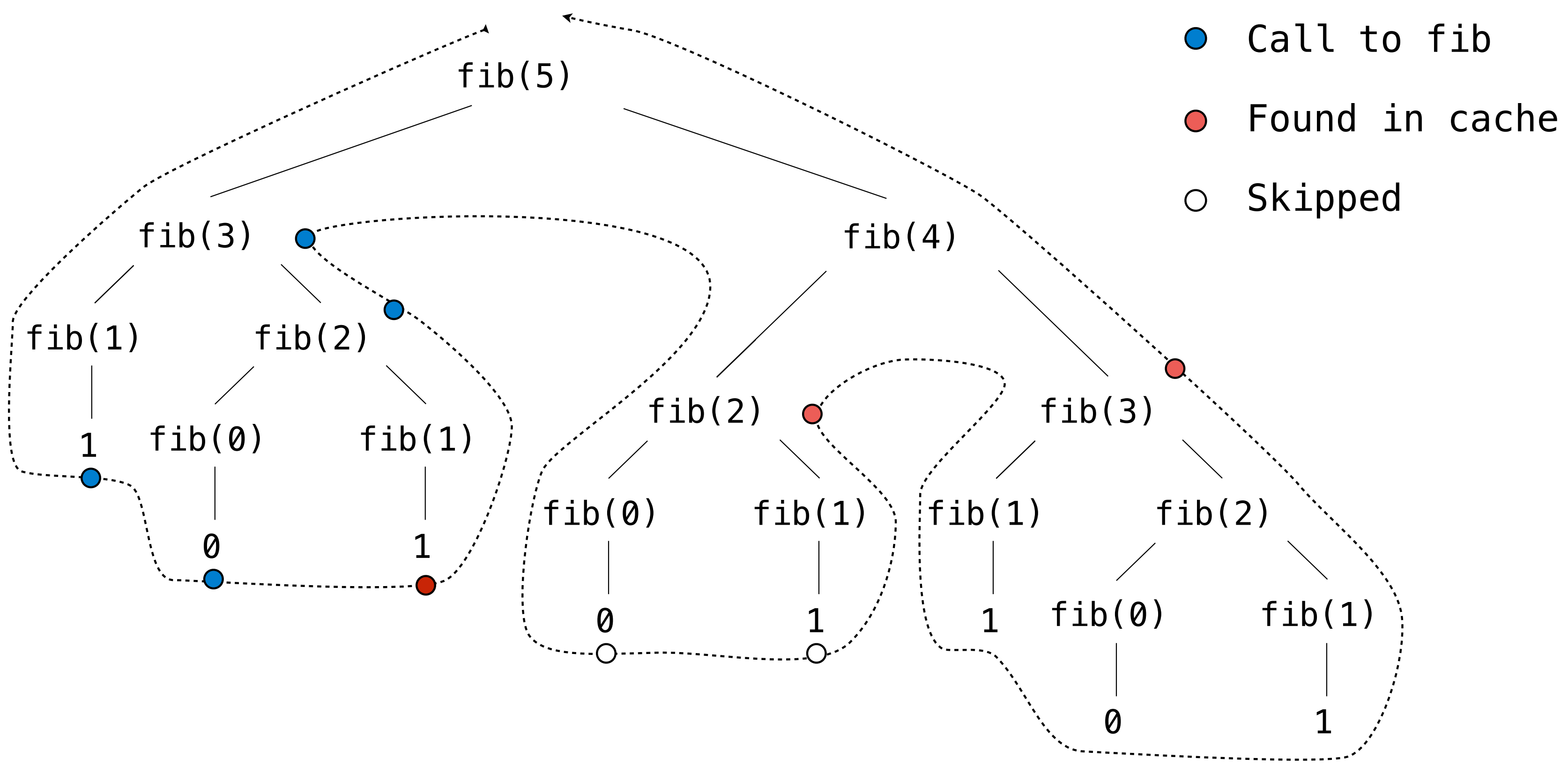

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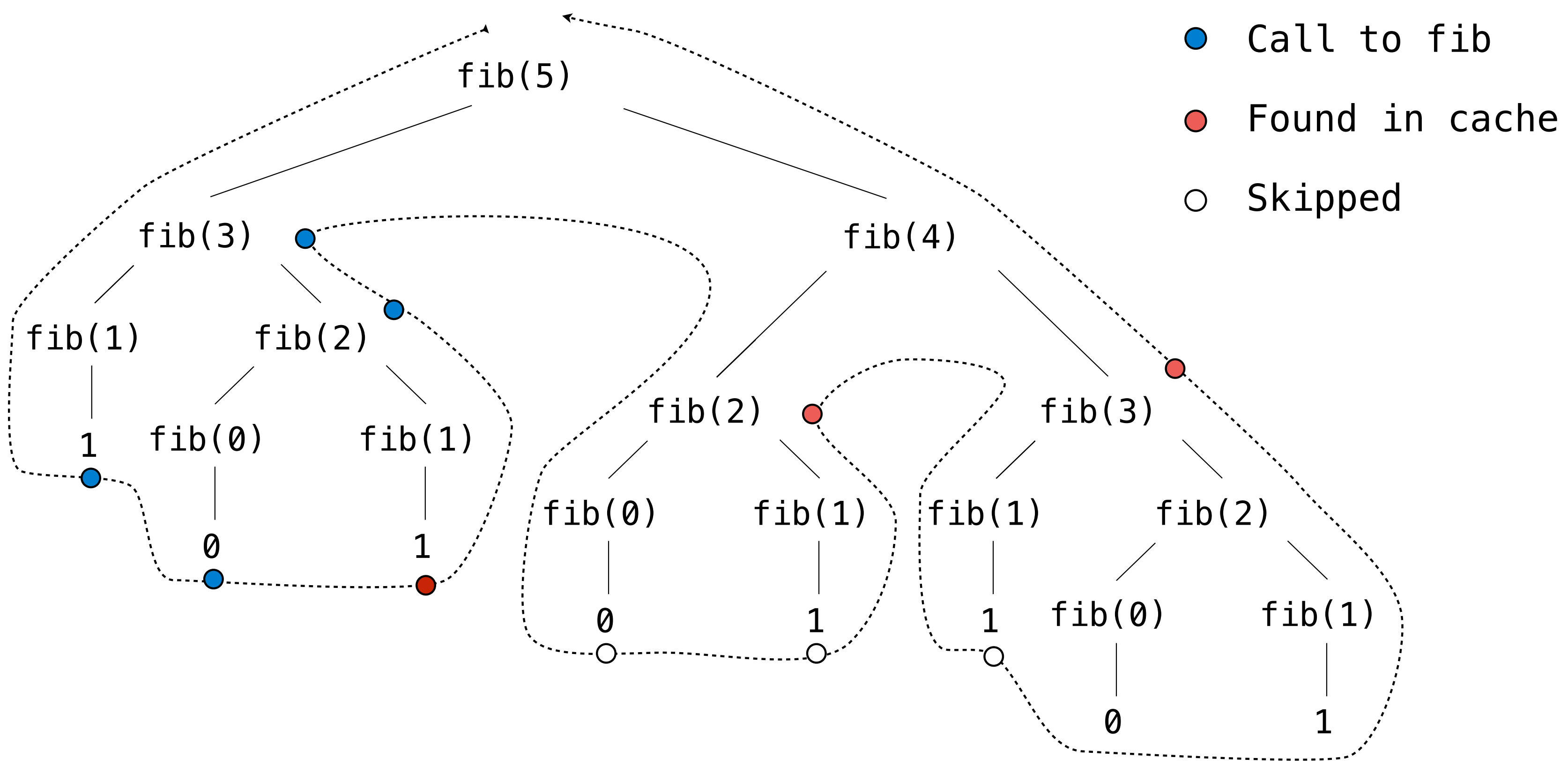

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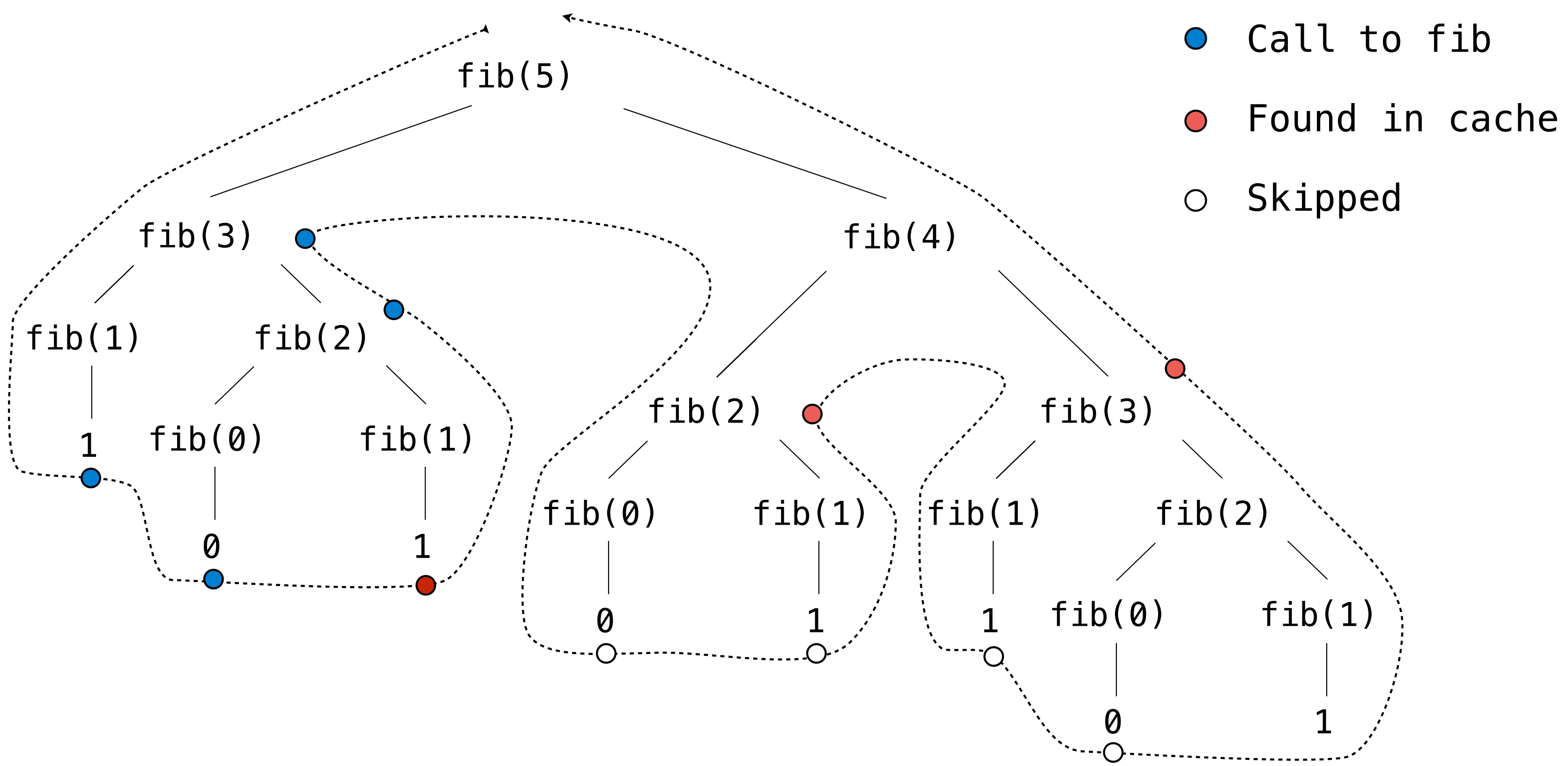

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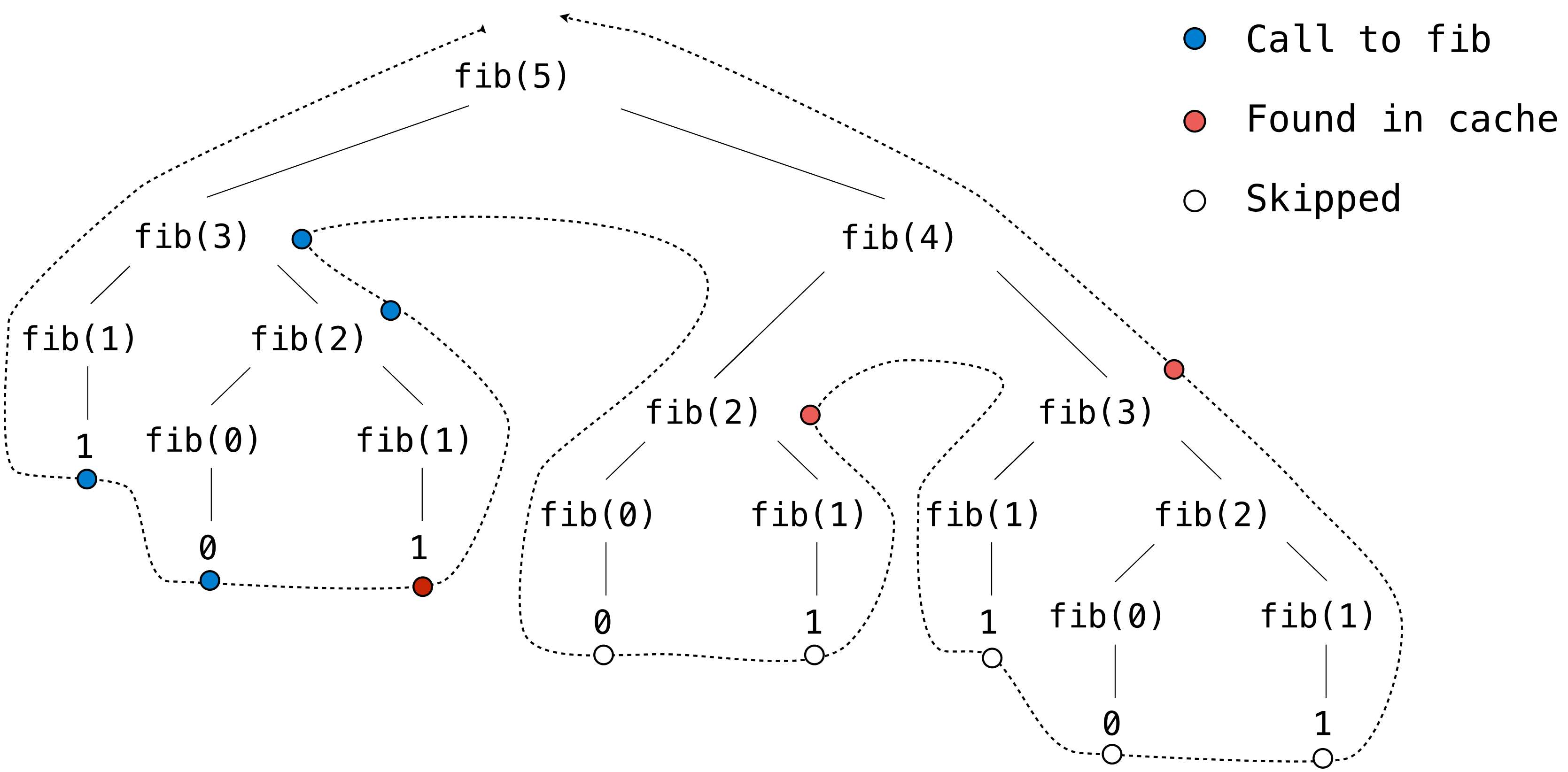

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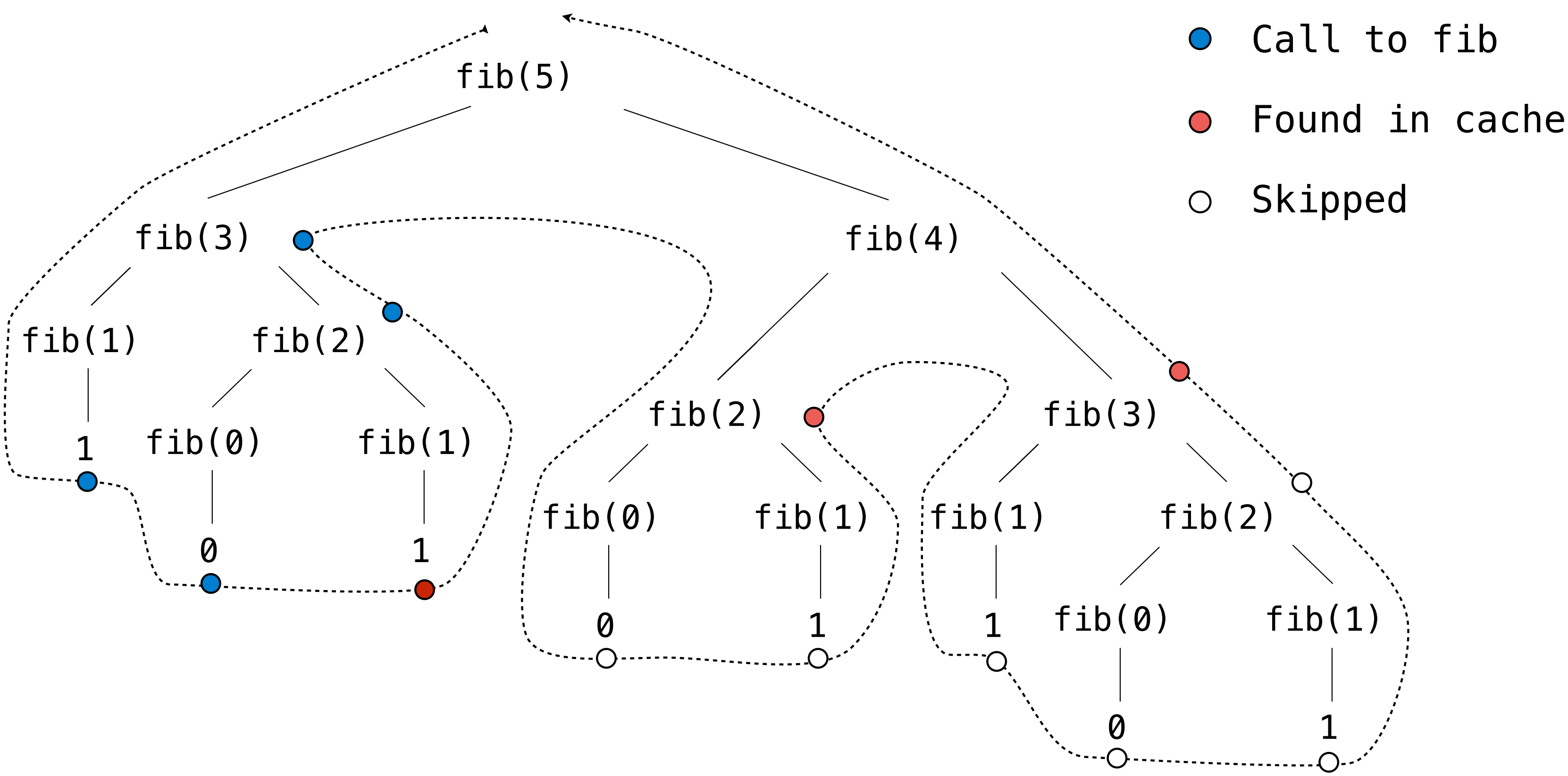

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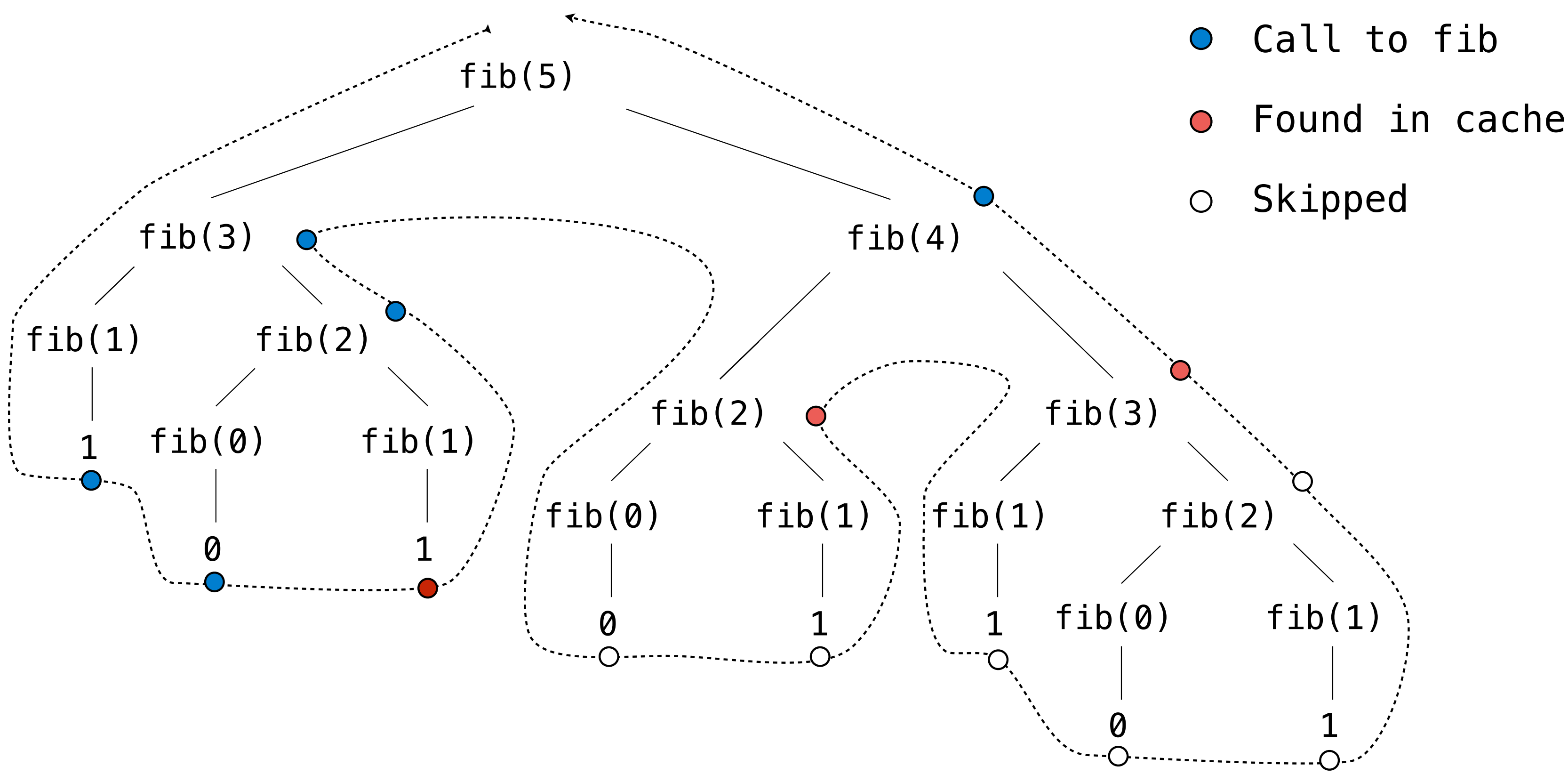

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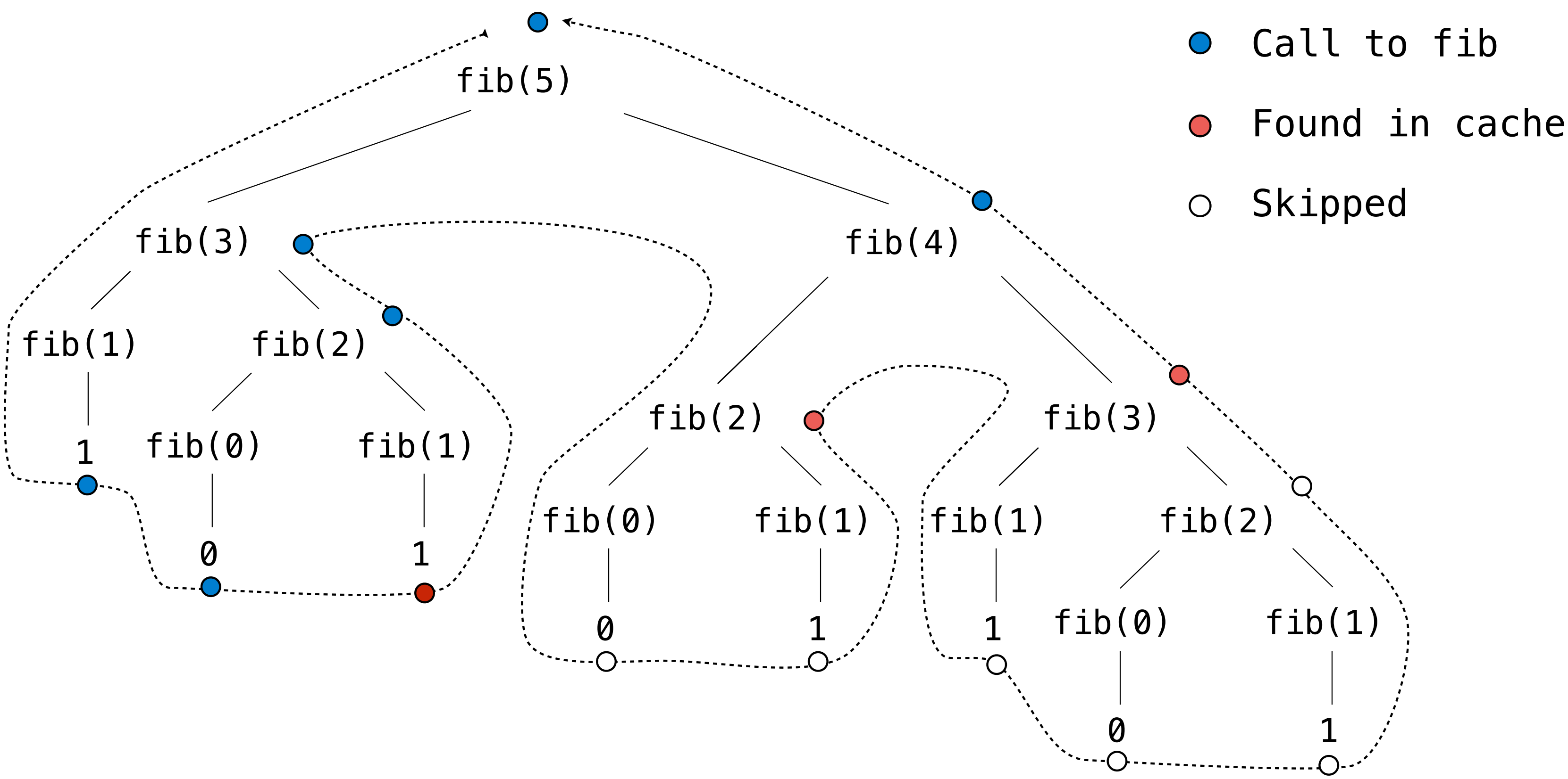

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```
def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)

def square(x):
    return x * x
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    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)

def square(x):
    return x * x
```

$$b^n = \begin{cases} 1 & \text{if } n = 0 \\ (b^{\frac{1}{2}n})^2 & \text{if } n \text{ is even} \\ b \cdot b^{n-1} & \text{if } n \text{ is odd} \end{cases}$$

(Demo)

Exponentiation

Goal: one more multiplication lets us double the problem size

```
def exp(b, n):
    if n == 0:
        return 1
    else:
        return b * exp(b, n-1)
```

```
def exp_fast(b, n):
    if n == 0:
        return 1
    elif n % 2 == 0:
        return square(exp_fast(b, n//2))
    else:
        return b * exp_fast(b, n-1)

def square(x):
    return x * x
```

Linear time:

- Doubling the input **doubles** the time
- 1024x the input takes 1024x as much time

Logarithmic time:

- Doubling the input **increases** the time by one step
- 1024x the input increases the time by only 10 steps

Orders of Growth

Quadratic Time

`Functions that process all pairs of values in a sequence of length n take quadratic time`

Quadratic Time

Functions that process all pairs of values in a sequence of length n take quadratic time

```
def overlap(a, b):
    count = 0
    for item in a:
        for other in b:
            if item == other:
                count += 1
    return count

overlap([3, 5, 7, 6], [4, 5, 6, 5])
```

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	3	5	7	6
4	0	0	0	0
5	0	1	0	0
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(Demo)

Exponential Time

Tree-recursive functions can take exponential time

```
def fib(n):
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return fib(n-2) + fib(n-1)
```

http://en.wikipedia.org/wiki/File:Fibonacci.jpg

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```

```
        fib(2)
       /      \
  fib(0)      fib(1)
    |           |
    0           1
```

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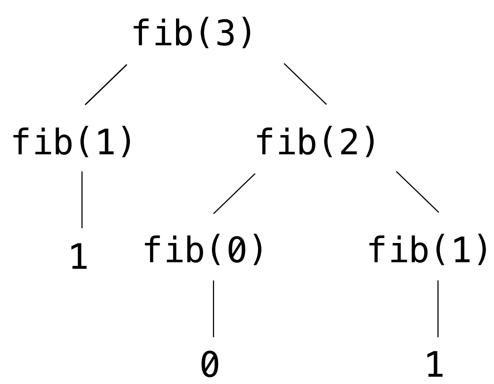

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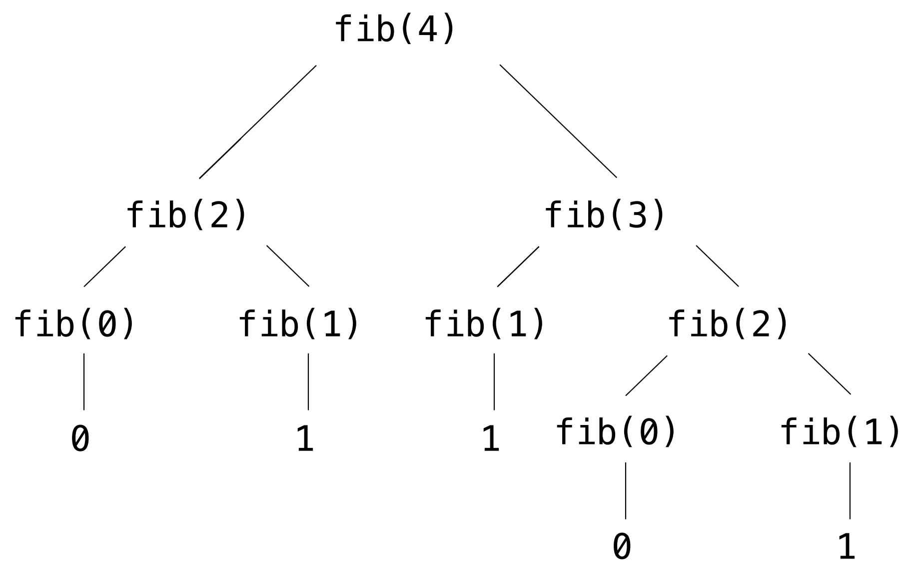

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```
                         fib(5)
              /                          \
         fib(3)                              fib(4)
        /      \                       /               \
   fib(1)      fib(2)              fib(2)                fib(3)
     |        /      \            /      \              /      \
     1    fib(0)    fib(1)    fib(0)    fib(1)     fib(1)      fib(2)
            |         |         |         |          |        /      \
            0         1         0         1          1    fib(0)    fib(1)
                                                            |         |
                                                            0         1
```

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Common Orders of Growth

Exponential growth. E.g., recursive `fib`

Quadratic growth. E.g., `overlap`

Linear growth. E.g., slow `exp`

Logarithmic growth. E.g., `exp_fast`

Constant growth. Increasing *n* doesn't affect time

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Order of Growth Notation

Big Theta and Big O Notation for Orders of Growth

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Linear growth. E.g., slow `exp` $\Theta(n)$

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Logarithmic growth. E.g., `exp_fast` $\Theta(\log n)$

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Big Theta and Big O Notation for Orders of Growth

Exponential growth. E.g., recursive `fib` Incrementing *n* multiplies *time* by a constant	$\Theta(b^n)$	$O(b^n)$
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Logarithmic growth. E.g., `exp_fast` Doubling *n* only increments *time* by a constant	$\Theta(\log n)$	$O(\log n)$
Constant growth. Increasing *n* doesn't affect time	$\Theta(1)$	$O(1)$

Space

Space and Environments

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Which environment frames do we need to keep during evaluation?

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(Demo)

pythontutor.com/composingprograms.html#code=def%20fib%28n%29%3A%0A%20%20%20%20if%20n%20%3D%3D%200%20or%20n%20%3D%3D%201%3A%0A%20%20%20%20%20%20%20%20return%20n%0A%20%20%20%20else%3A%0A%20%20%20%20%20%20%20%20return%20fib%28n-2%29%20%2B%20fib%28n-1%29%0A%20%20%20%20%20%20%20%20%0Afib%286%29&mode=display&origin=composingprograms.js&cumulative=false&py=3&rawInputLstJSON=[]&curInstr=1

Fibonacci Space Consumption

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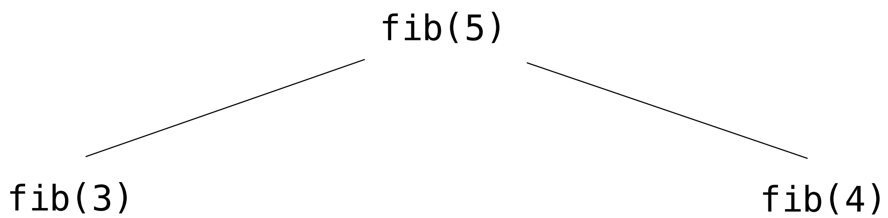
```
fib(5)
```

Fibonacci Space Consumption

```
              fib(5)
             /
       fib(3)
```

Fibonacci Space Consumption



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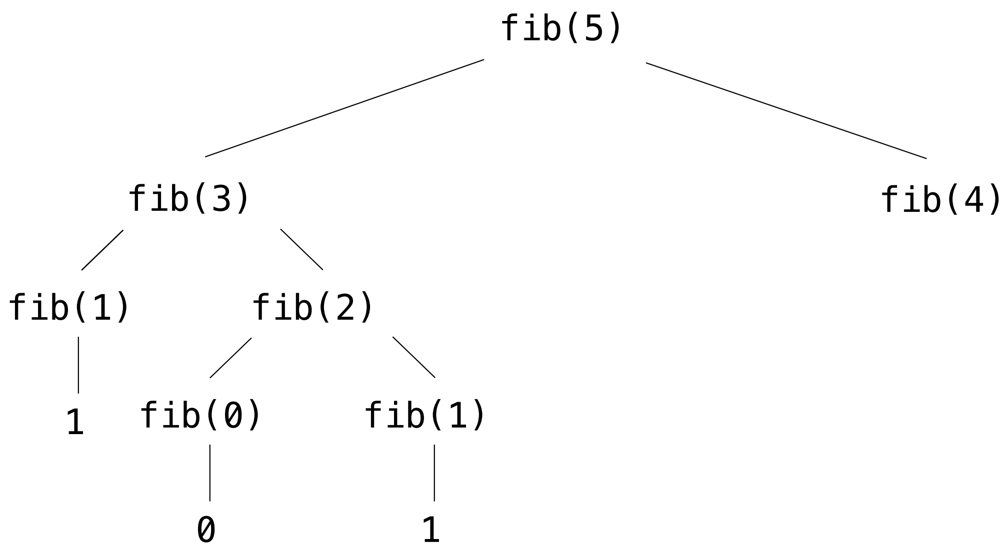

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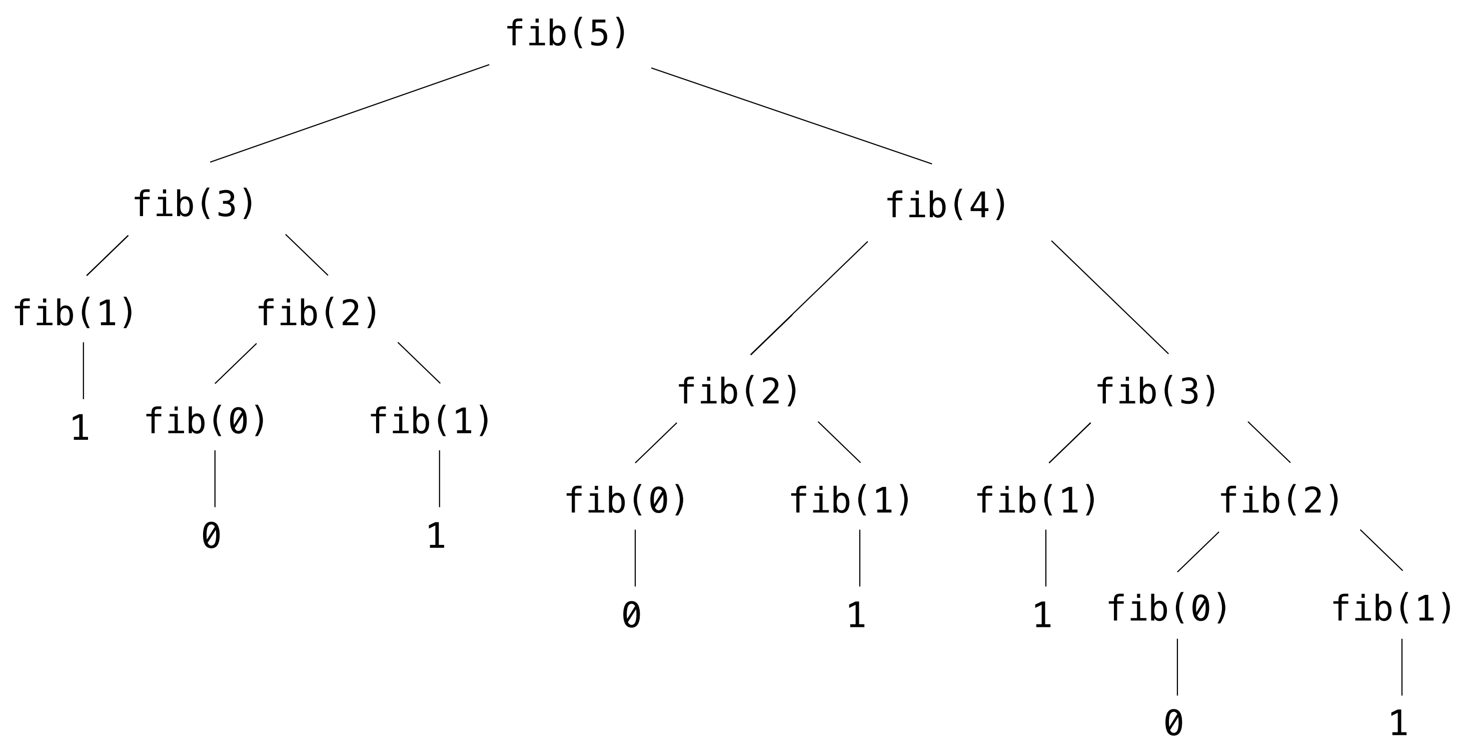

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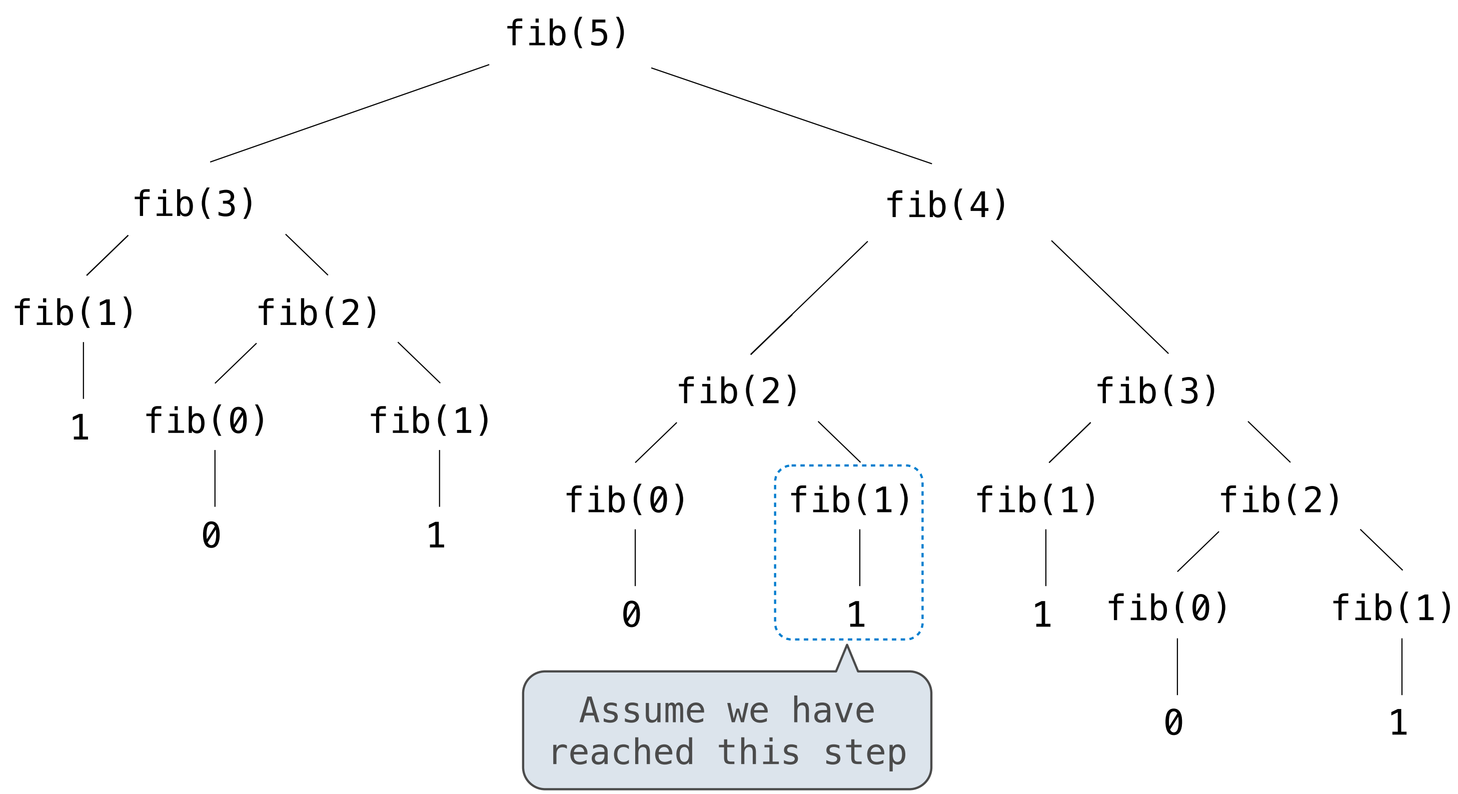

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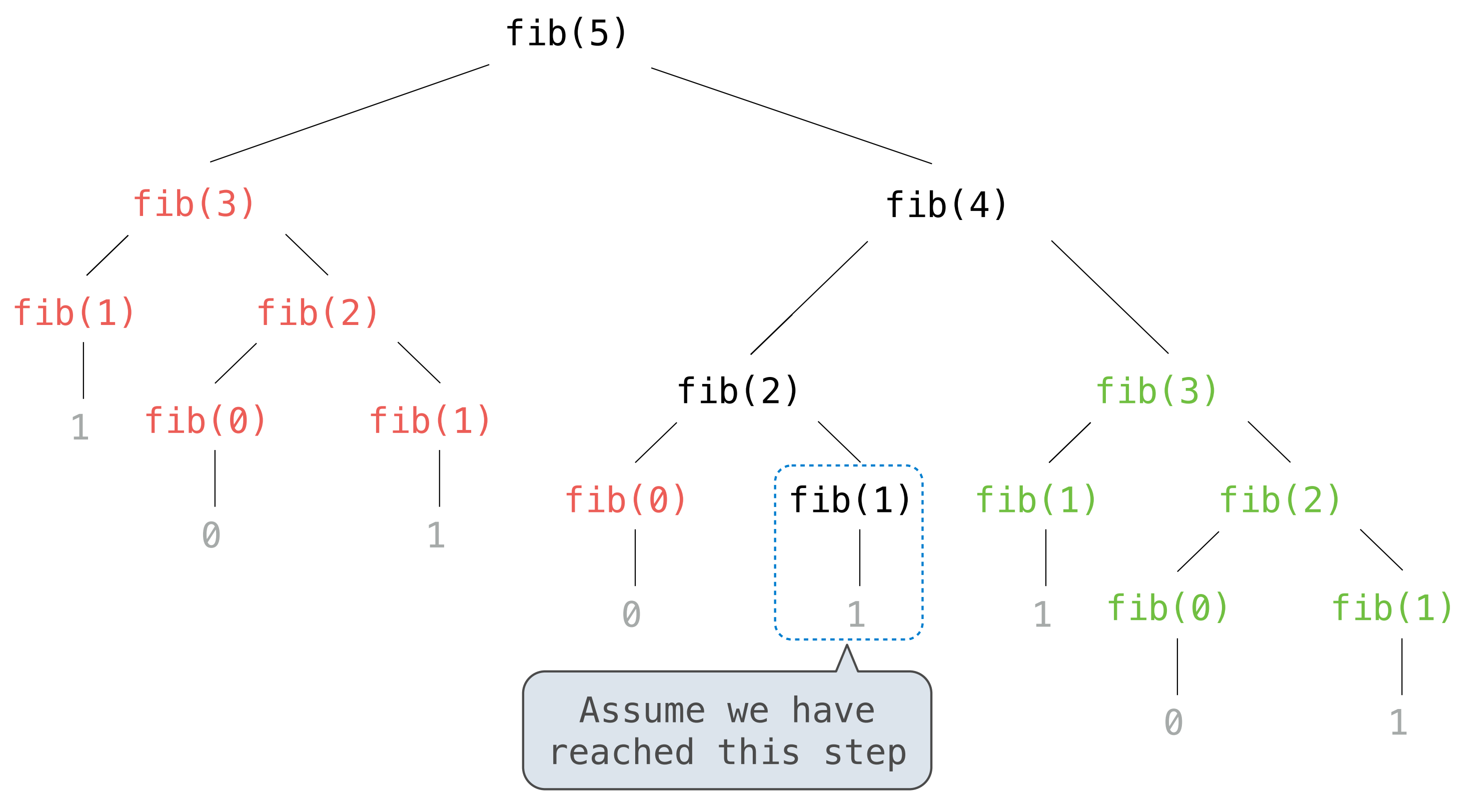

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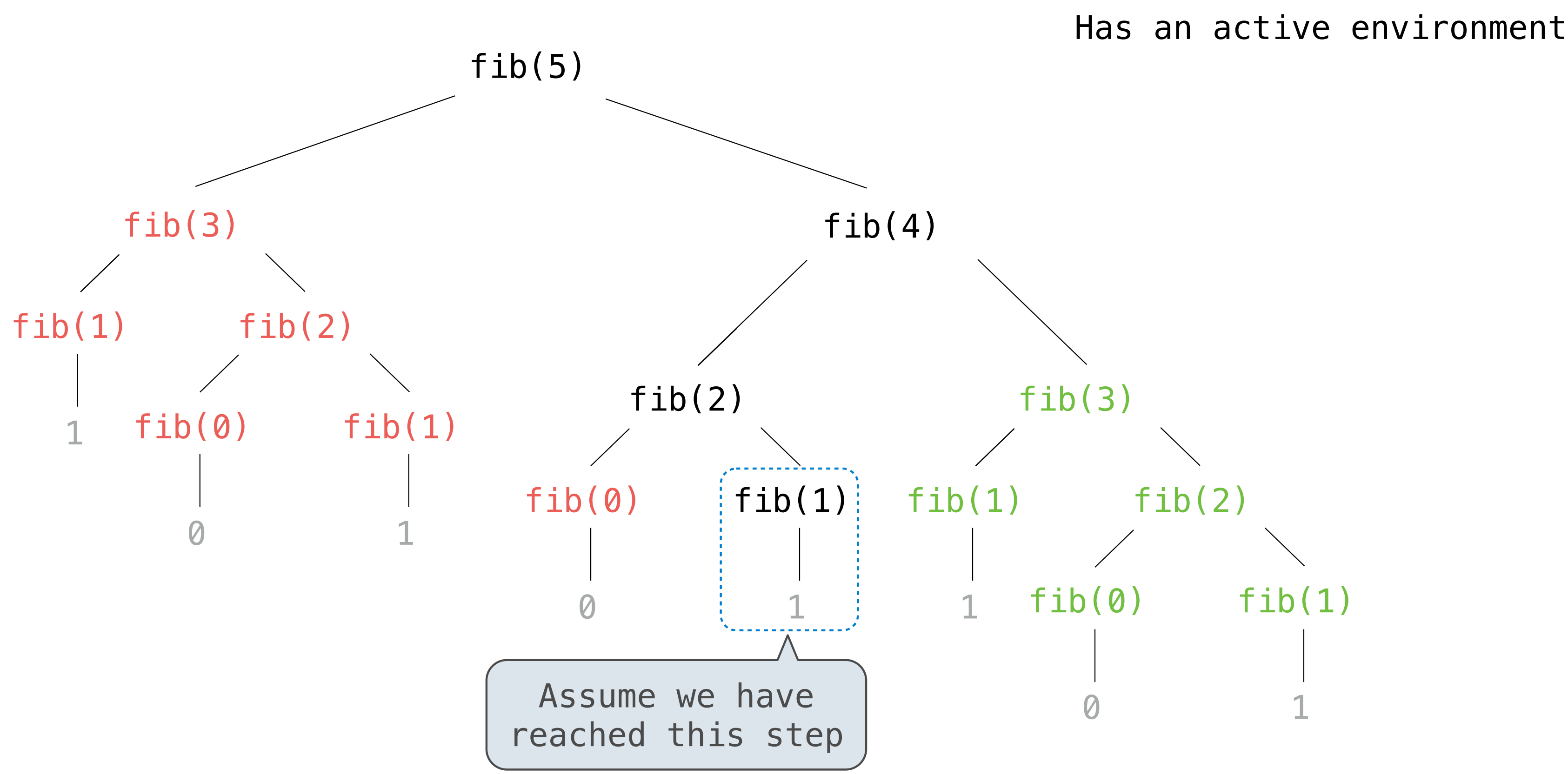

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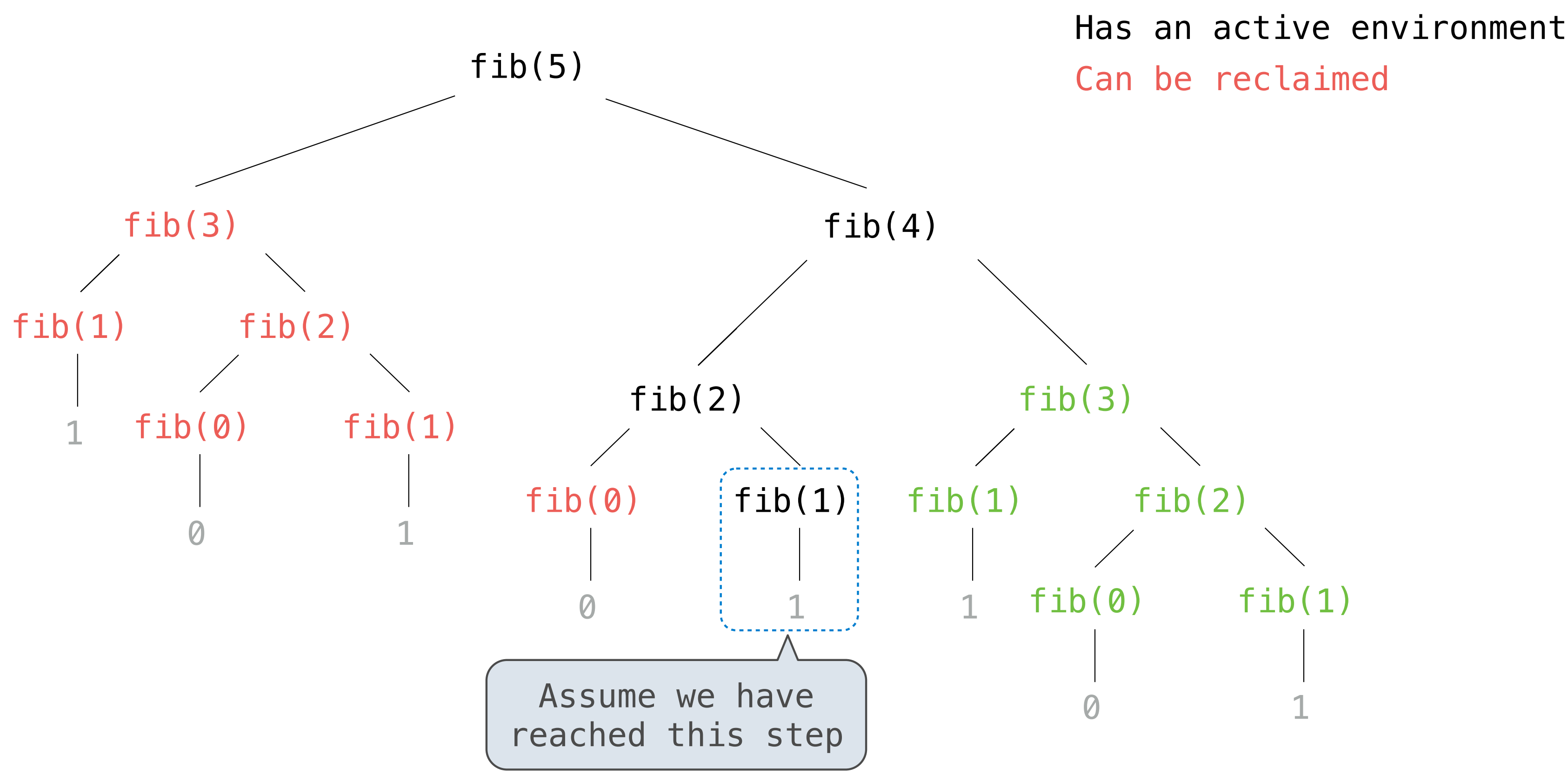

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