Higher-Order Functions

Announcements

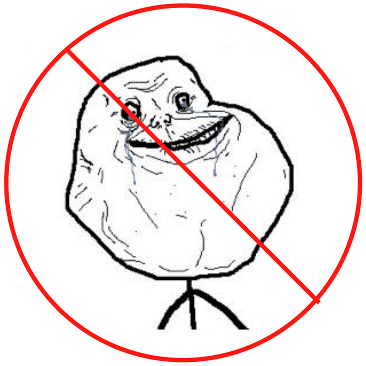
Office Hours: You Should Go!

Office Hours: You Should Go!

You are not alone!

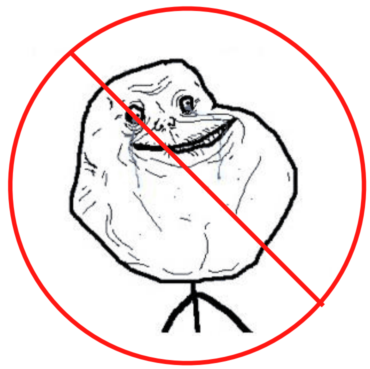
Office Hours: You Should Go!

You are not alone!



Office Hours: You Should Go!

You are not alone!



https://cs61a.org/office-hours/

Example: Prime Factorization

Prime Factorization

Prime Factorization

Each positive integer n has a set of prime factors: primes whose product is n

Prime Factorization

```
Each positive integer n has a set of prime factors: primes whose product is n

...
8  = 2 * 2 * 2
9  = 3 * 3
10 = 2 * 5
11 = 11
12 = 2 * 2 * 3
...
```

Prime Factorization

Each positive integer n has a set of prime factors: primes whose product is n

```
...
8  = 2 * 2 * 2
9  = 3 * 3
10 = 2 * 5
11 = 11
12 = 2 * 2 * 3
...
```

One approach: Find the smallest prime factor of n, then divide by it

Prime Factorization

Each positive integer n has a set of prime factors: primes whose product is n

```
...
8  = 2 * 2 * 2
9  = 3 * 3
10 = 2 * 5
11 = 11
12 = 2 * 2 * 3
...
```

One approach: Find the smallest prime factor of n, then divide by it

858

Prime Factorization

Each positive integer n has a set of prime factors: primes whose product is n

```
...
8  = 2 * 2 * 2
9  = 3 * 3
10 = 2 * 5
11 = 11
12 = 2 * 2 * 3
...
```

One approach: Find the smallest prime factor of n, then divide by it

```
858  = 2 * 429
```

Prime Factorization

Each positive integer n has a set of prime factors: primes whose product is n

```
...
8  = 2 * 2 * 2
9  = 3 * 3
10 = 2 * 5
11 = 11
12 = 2 * 2 * 3
...
```

One approach: Find the smallest prime factor of n, then divide by it

```
858  = 2 * 429  = 2 * 3 * 143
```

Prime Factorization

Each positive integer n has a set of prime factors: primes whose product is n

```
...
8  = 2 * 2 * 2
9  = 3 * 3
10 = 2 * 5
11 = 11
12 = 2 * 2 * 3
...
```

One approach: Find the smallest prime factor of n, then divide by it

858 = 2 * 429 = 2 * 3 * 143 = 2 * 3 * 11 * 13

Prime Factorization

Each positive integer n has a set of prime factors: primes whose product is n

```
...
8  = 2 * 2 * 2
9  = 3 * 3
10 = 2 * 5
11 = 11
12 = 2 * 2 * 3
...
```

One approach: Find the smallest prime factor of n, then divide by it

858 = 2 * 429 = 2 * 3 * 143 = 2 * 3 * 11 * 13

(Demo)

Example: Iteration

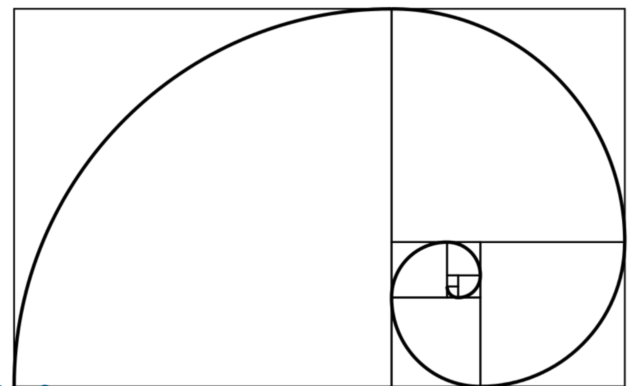
The Fibonacci Sequence

The Fibonacci Sequence

The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

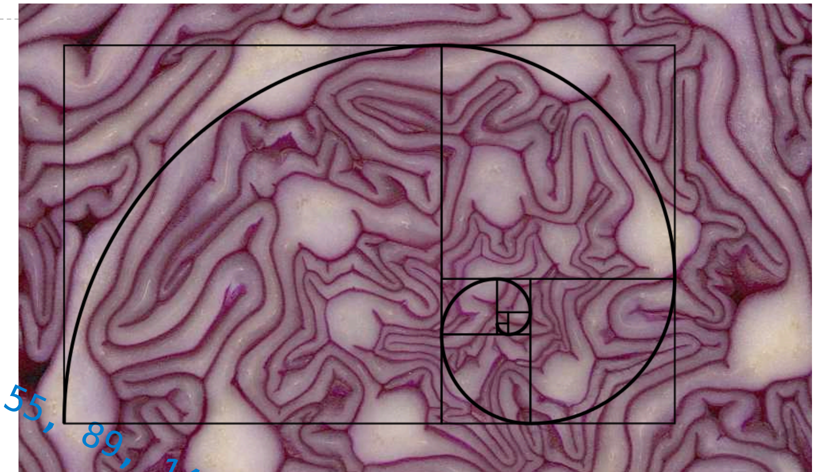
The Fibonacci Sequence



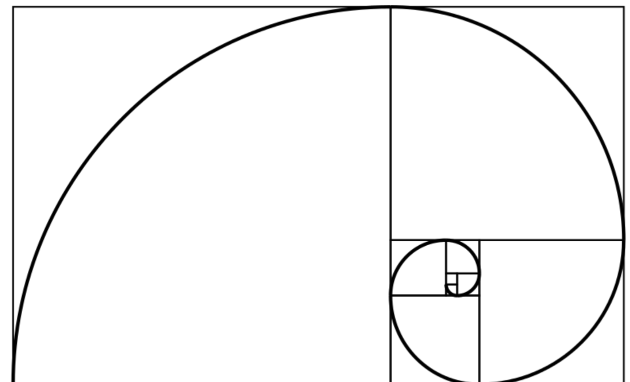
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987



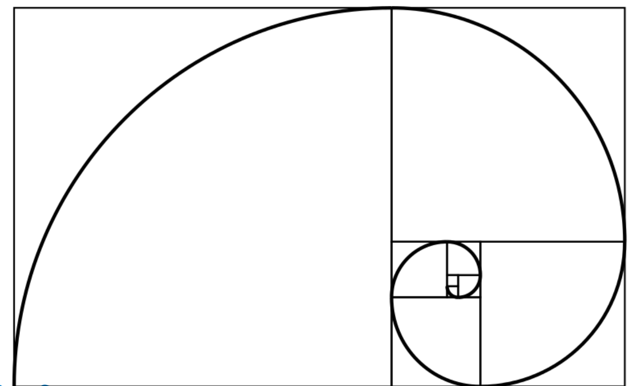
The Fibonacci Sequence



0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

The Fibonacci Sequence

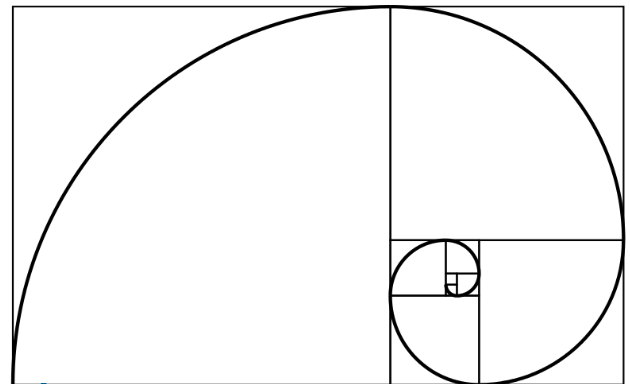
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987



```
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987



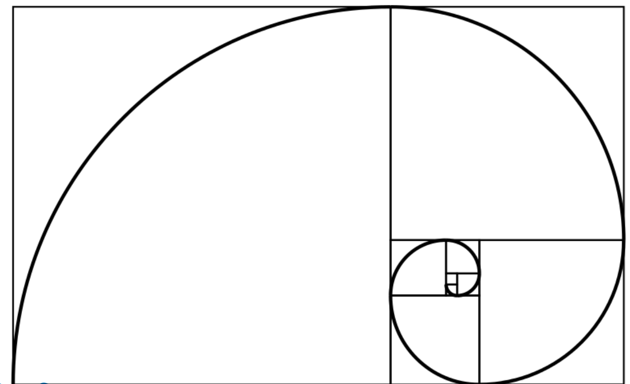
```
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor

The Fibonacci Sequence

fib pred
curr
n 5
k

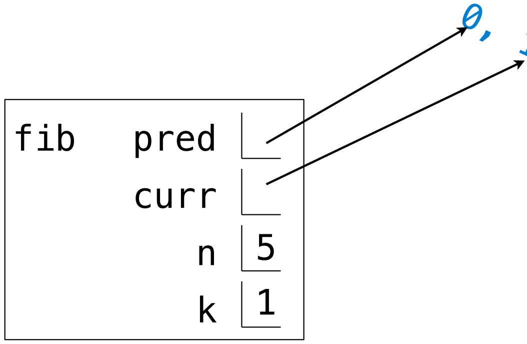
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987



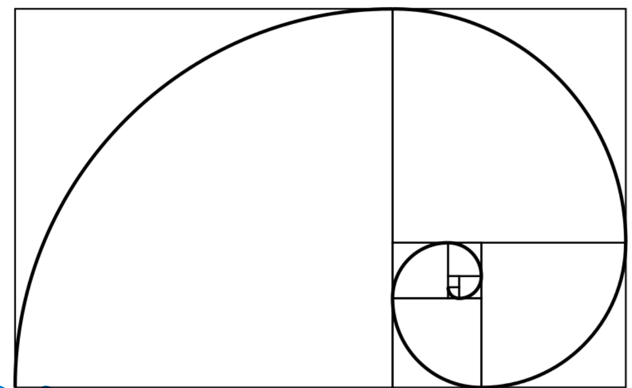
```
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor

The Fibonacci Sequence



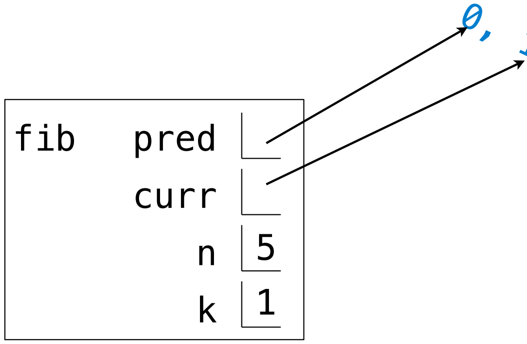
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987



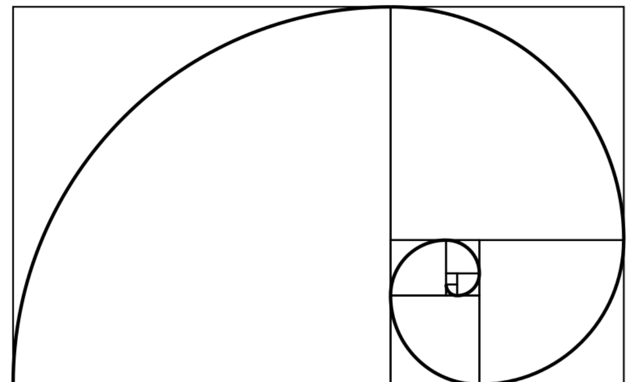
```
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor

The Fibonacci Sequence



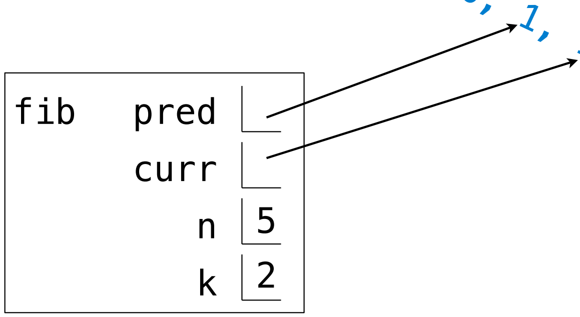
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987



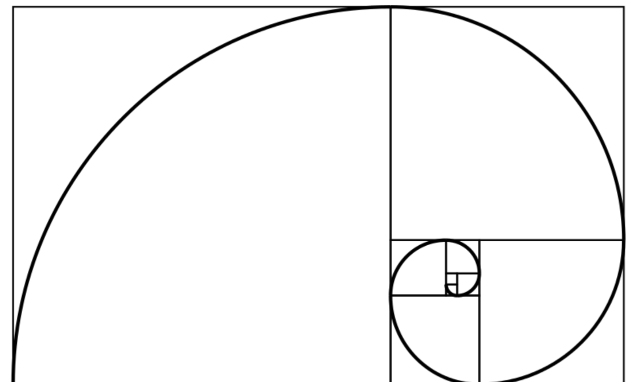
```python
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor

The Fibonacci Sequence



0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

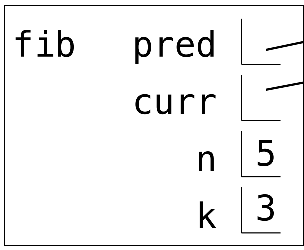


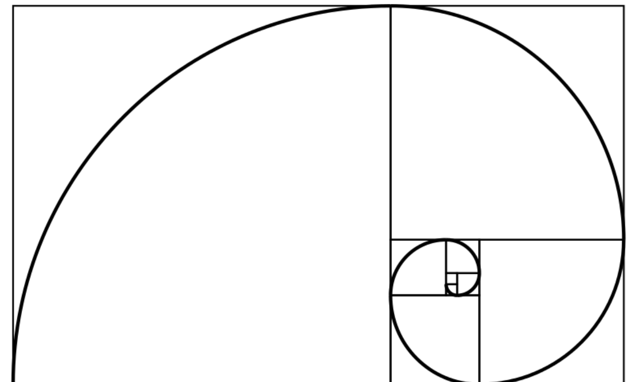
```
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor

The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

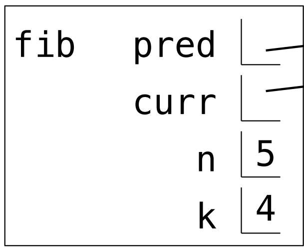




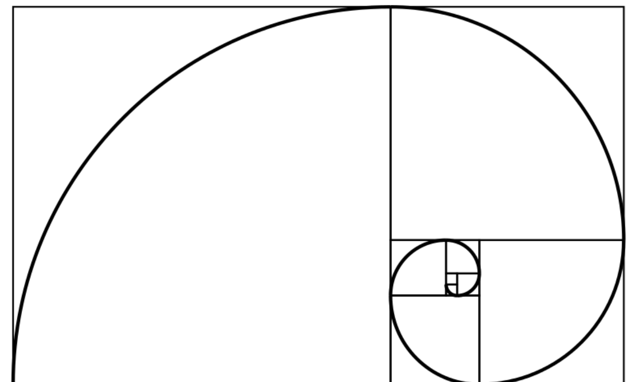
```python
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor

The Fibonacci Sequence



0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

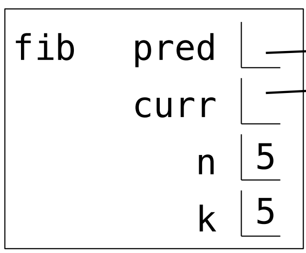


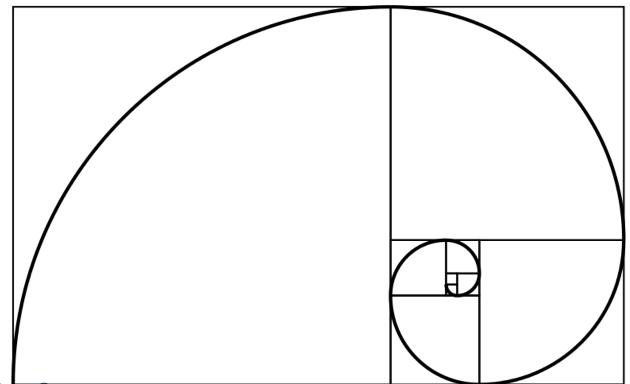
```
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor

The Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987

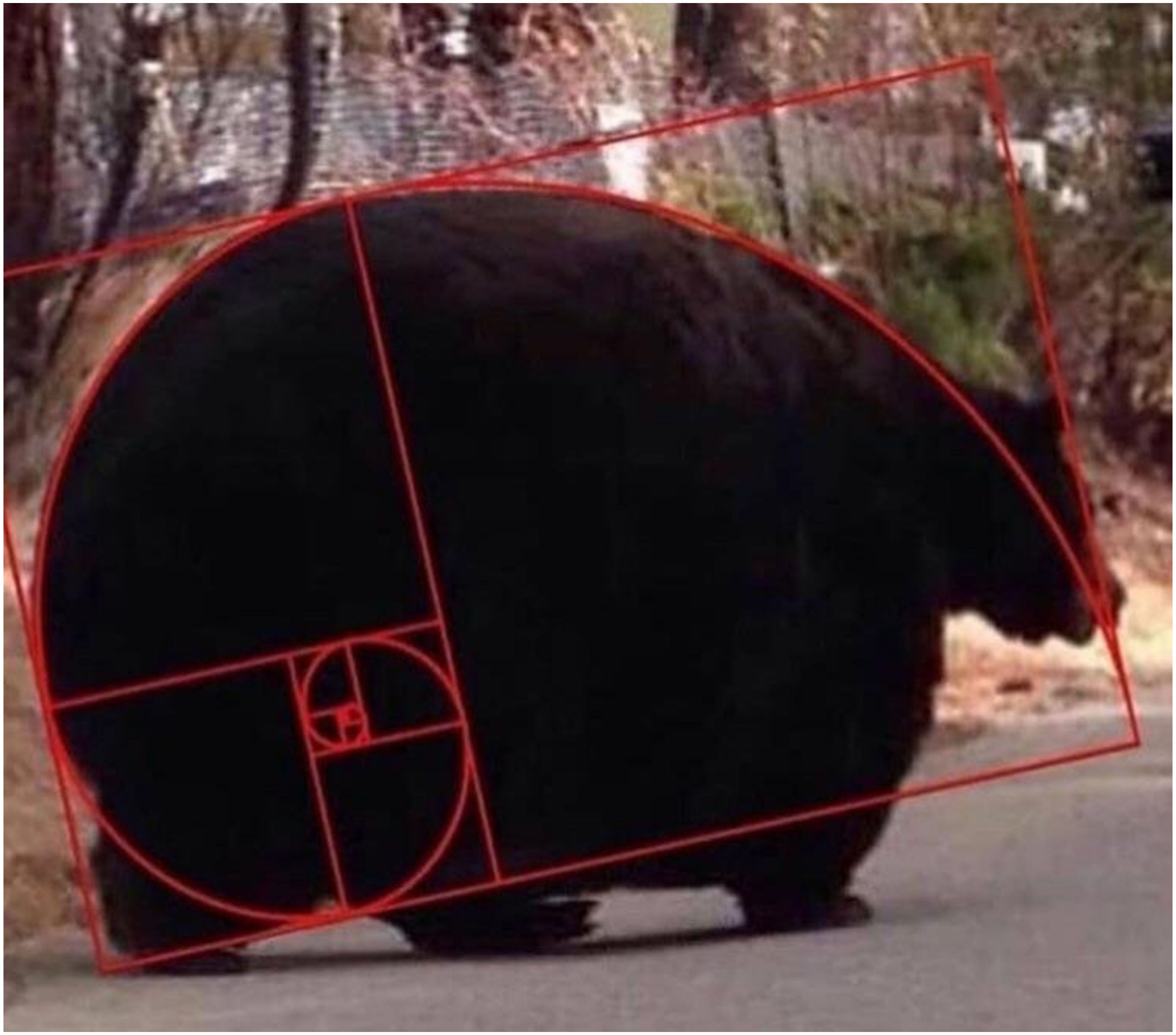




```
def fib(n):
    """Compute the nth Fibonacci number, for N >= 1."""
    pred, curr = 0, 1   # 0th and 1st Fibonacci numbers
    k = 1               # curr is the kth Fibonacci number
    while k < n:
        pred, curr = curr, pred + curr
        k = k + 1
    return curr
```

The next Fibonacci number is the sum of the current one and its predecessor

Go Bears!



Designing Functions

Describing Functions

Describing Functions

A function's *domain* is the set of all inputs it might possibly take as arguments.

Describing Functions

A function's *domain* is the set of all inputs it might possibly take as arguments.

A function's *range* is the set of output values it might possibly return.

Describing Functions

A function's *domain* is the set of all inputs it might possibly take as arguments.

A function's *range* is the set of output values it might possibly return.

A pure function's *behavior* is the relationship it creates between input and output.

Describing Functions

```
def square(x):
    """Return X * X."""
```

A function's *domain* is the set of all inputs it might possibly take as arguments.

A function's *range* is the set of output values it might possibly return.

A pure function's *behavior* is the relationship it creates between input and output.

Describing Functions

A function's *domain* is the set of all inputs it might possibly take as arguments.

A function's *range* is the set of output values it might possibly return.

A pure function's *behavior* is the relationship it creates between input and output.

```
def square(x):
    """Return X * X."""
```

x is a number

Describing Functions

```
def square(x):
    """Return X * X."""
```

A function's *domain* is the set of all inputs it might possibly take as arguments.

x is a number

A function's *range* is the set of output values it might possibly return.

square returns a non-negative real number

A pure function's *behavior* is the relationship it creates between input and output.

Describing Functions

```
def square(x):
    """Return X * X."""
```

A function's *domain* is the set of all inputs it might possibly take as arguments.

x is a number

A function's *range* is the set of output values it might possibly return.

square returns a non-negative real number

A pure function's *behavior* is the relationship it creates between input and output.

square returns the square of x

A Guide to Designing Function

A Guide to Designing Function

```
Give each function exactly one job, but make it apply to many related situations
```

A Guide to Designing Function

Give each function exactly one job, but make it apply to many related situations

```
>>> round(1.23)
1
```

A Guide to Designing Function

Give each function exactly one job, but make it apply to many related situations

```
>>> round(1.23)        >>> round(1.23, 1)
1                      1.2
```

A Guide to Designing Function

Give each function exactly one job, but make it apply to many related situations

```
>>> round(1.23)
1
```

```
>>> round(1.23, 1)
1.2
```

```
>>> round(1.23, 0)
1
```

A Guide to Designing Function

Give each function exactly one job, but make it apply to many related situations

```
>>> round(1.23)
1
```

```
>>> round(1.23, 1)
1.2
```

```
>>> round(1.23, 0)
1
```

```
>>> round(1.23, 5)
1.23
```

A Guide to Designing Function

Give each function exactly one job, but make it apply to many related situations

```
>>> round(1.23)
1
```

```
>>> round(1.23, 1)
1.2
```

```
>>> round(1.23, 0)
1
```

```
>>> round(1.23, 5)
1.23
```

Don’t repeat yourself (DRY): Implement a process just once, but execute it many times

A Guide to Designing Function

Give each function exactly one job, but make it apply to many related situations

```
>>> round(1.23)
1
```

```
>>> round(1.23, 1)
1.2
```

```
>>> round(1.23, 0)
1
```

```
>>> round(1.23, 5)
1.23
```

Don’t repeat yourself (DRY): Implement a process just once, but execute it many times

(Demo)

Generalization

Generalizing Patterns with Arguments

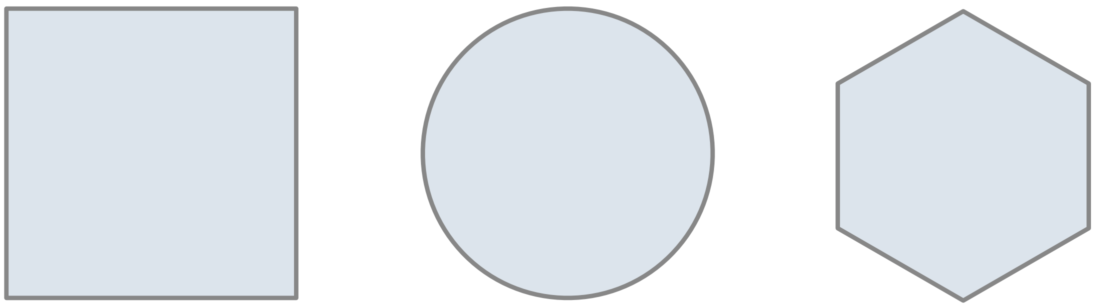
Generalizing Patterns with Arguments

```
Regular geometric shapes relate length and area.
```

Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

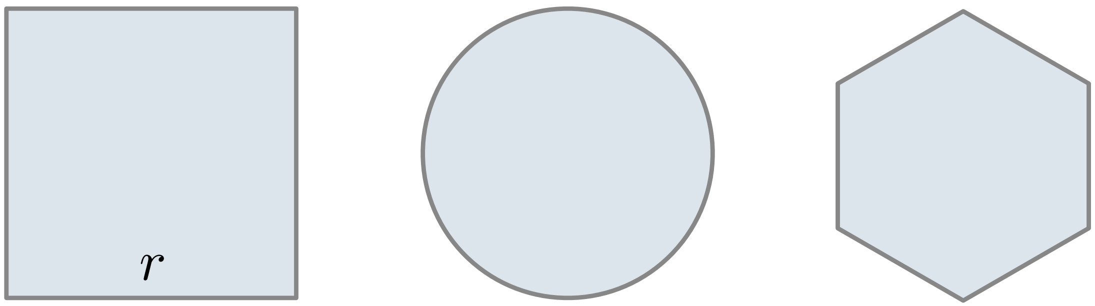
Shape:



Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

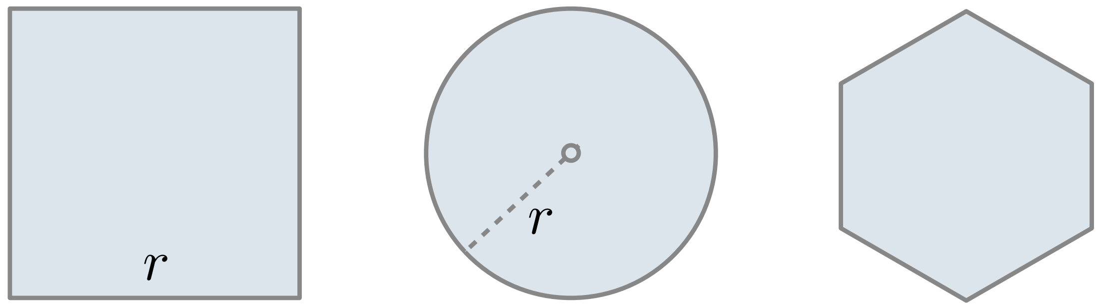
Shape:



Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

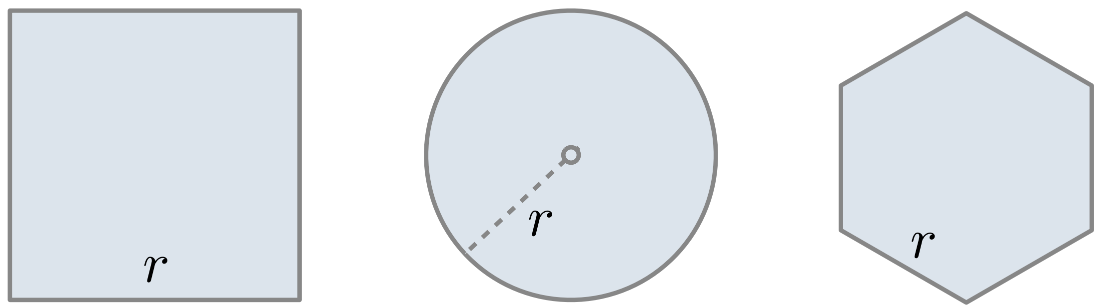
Shape:



Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

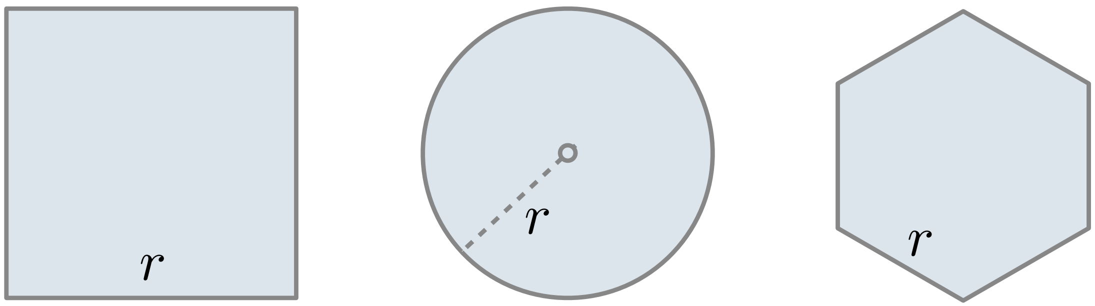
Shape:



Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

Shape:

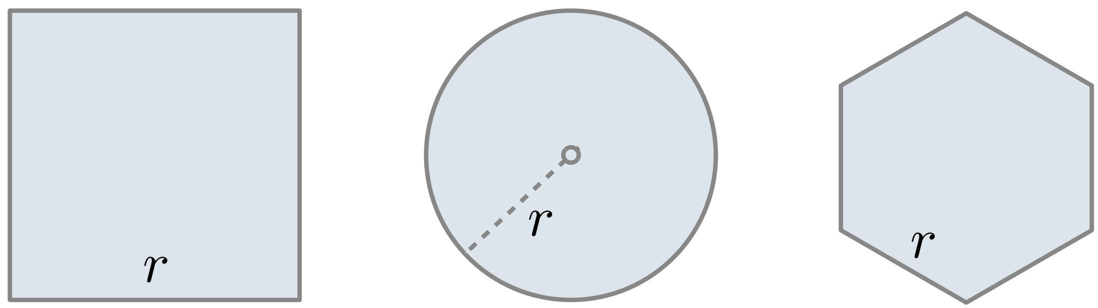


Area:

Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

Shape:

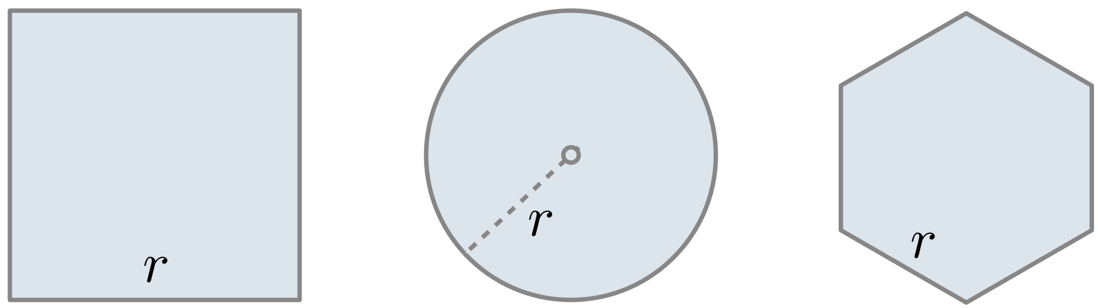


Area: r^2

Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

Shape:

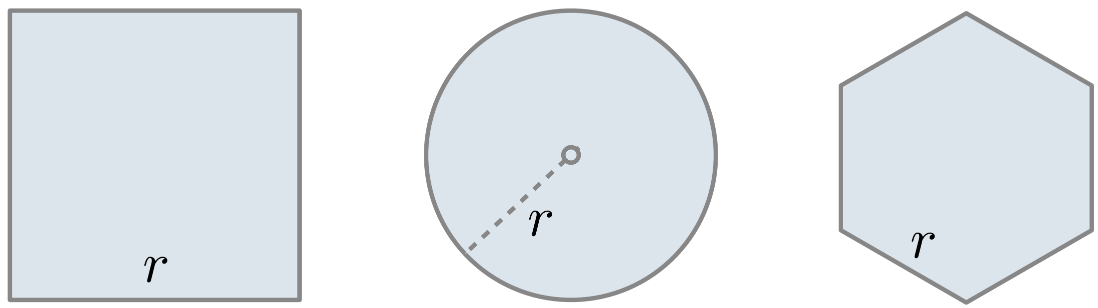


Area: r^2 $\pi \cdot r^2$

Generalizing Patterns with Arguments

`Regular geometric shapes relate length and area.`

Shape:



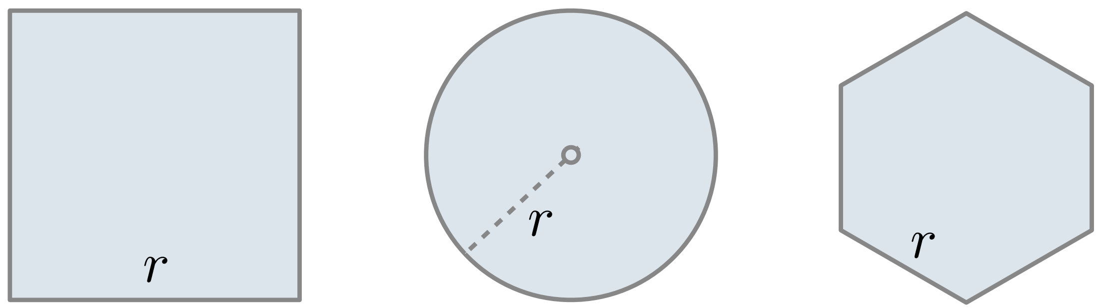
Area:

$$r^2 \qquad \pi \cdot r^2 \qquad \frac{3\sqrt{3}}{2} \cdot r^2$$

Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

Shape:



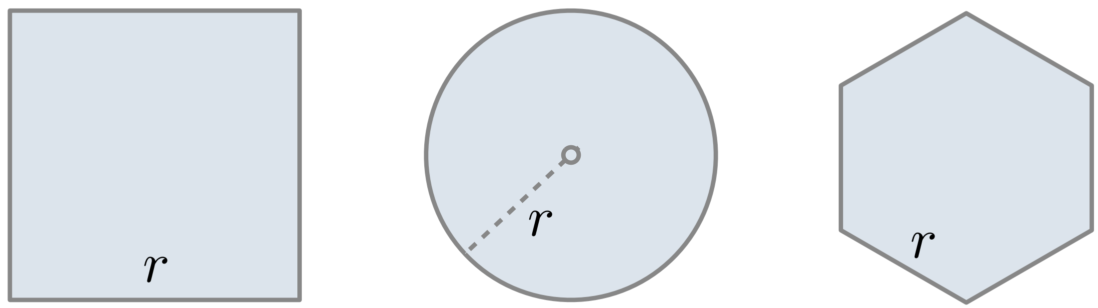
Area:

$1 \cdot r^2$ $\quad$ $\pi \cdot r^2$ $\quad$ $\frac{3\sqrt{3}}{2} \cdot r^2$

Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

Shape:

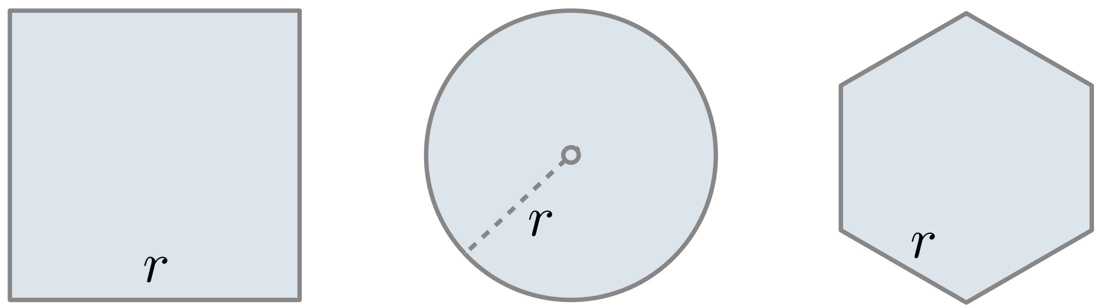


Area: $1 \cdot r^2$ $\pi \cdot r^2$ $\frac{3\sqrt{3}}{2} \cdot r^2$

Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

Shape:



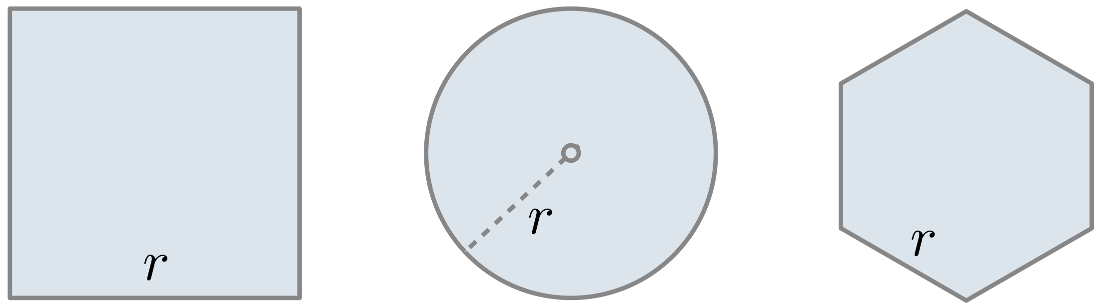
Area:

$1 \cdot r^2$ $\pi \cdot r^2$ $\frac{3\sqrt{3}}{2} \cdot r^2$

Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

Shape:



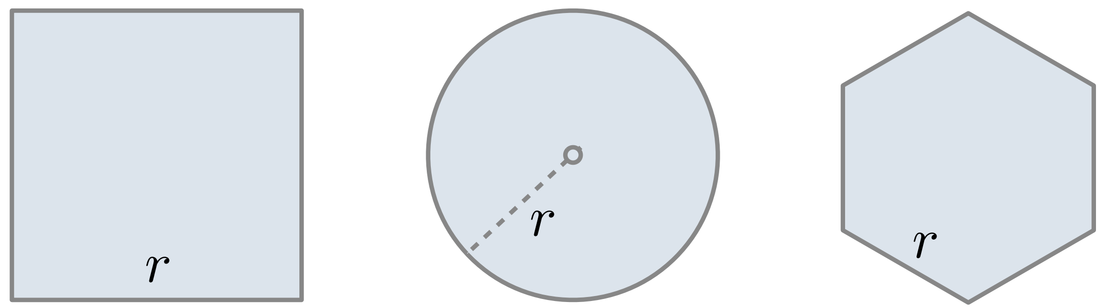
Area:

$1 \cdot r^2$ $\pi \cdot r^2$ $\frac{3\sqrt{3}}{2} \cdot r^2$

Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

Shape:



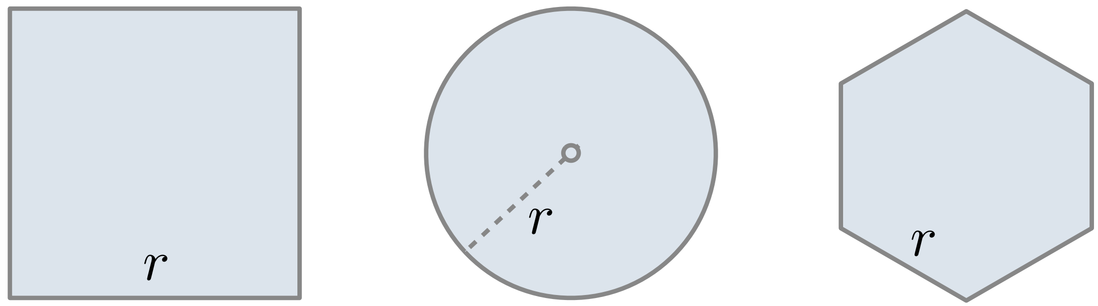
Area: $1 \cdot r^2$ $\qquad \pi \cdot r^2$ $\qquad \frac{3\sqrt{3}}{2} \cdot r^2$

Finding common structure allows for shared implementation

Generalizing Patterns with Arguments

Regular geometric shapes relate length and area.

Shape:



Area: $1 \cdot r^2$ $\pi \cdot r^2$ $\frac{3\sqrt{3}}{2} \cdot r^2$

Finding common structure allows for shared implementation

(Demo)

Higher-Order Functions

Generalizing Over Computational Processes

Generalizing Over Computational Processes

```
The common structure among functions may be a computational process, rather than a number.
```

Generalizing Over Computational Processes

The common structure among functions may be a computational process, rather than a number.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 \qquad = 15$$

$$\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \qquad = 225$$

$$\sum_{k=1}^{5} \frac{8}{(4k-3)\cdot(4k-1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} \qquad = 3.04$$

Generalizing Over Computational Processes

The common structure among functions may be a computational process, rather than a number.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 \qquad = 15$$

$$\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \qquad = 225$$

$$\sum_{k=1}^{5} \frac{8}{(4k-3)\cdot(4k-1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} \qquad = 3.04$$

Generalizing Over Computational Processes

The common structure among functions may be a computational process, rather than a number.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 \qquad = 15$$

$$\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \qquad = 225$$

$$\sum_{k=1}^{5} \frac{8}{(4k-3)\cdot(4k-1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} \qquad = 3.04$$

Generalizing Over Computational Processes

```
The common structure among functions may be a computational process, rather than a number.
```

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$$

$$\sum_{k=1}^{5} \frac{8}{(4k-3)\cdot(4k-1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} = 3.04$$

Generalizing Over Computational Processes

The common structure among functions may be a computational process, rather than a number.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 \qquad = 15$$

$$\sum_{k=1}^{5} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \qquad = 225$$

$$\sum_{k=1}^{5} \frac{8}{(4k-3)\cdot(4k-1)} = \frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} \qquad = 3.04$$

(Demo)

Summation Example

```python
def cube(k):
    return pow(k, 3)

def summation(n, term):
    """Sum the first n terms of a sequence.

    >>> summation(5, cube)
    225
    """
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k + 1
    return total
```

Summation Example

```
def cube(k):
    return pow(k, 3)
```

Function of a single argument (*not called "term"*)

```
def summation(n, term):
    """Sum the first n terms of a sequence.

    >>> summation(5, cube)
    225
    """
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k + 1
    return total
```

Summation Example

```
def cube(k):
    return pow(k, 3)

def summation(n, term):
    """Sum the first n terms of a sequence.

    >>> summation(5, cube)
    225
    """
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k + 1
    return total
```

Function of a single argument (*not called "term"*)

A formal parameter that will be bound to a function

Summation Example

```
def cube(k):
    return pow(k, 3)

def summation(n, term):
    """Sum the first n terms of a sequence.

    >>> summation(5, cube)
    225
    """
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k + 1
    return total
```

Function of a single argument (*not called "term"*)

A formal parameter that will be bound to a function

The function bound to term gets called here

Summation Example

```
def cube(k):
    return pow(k, 3)

def summation(n, term):
    """Sum the first n terms of a sequence.

    >>> summation(5, cube)
    225
    """
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k + 1
    return total
```

Function of a single argument (*not called "term"*)

A formal parameter that will be bound to a function

The cube function is passed as an argument value

The function bound to term gets called here

Summation Example

```
def cube(k):
    return pow(k, 3)

def summation(n, term):
    """Sum the first n terms of a sequence.

    >>> summation(5, cube)
    225
    """
    total, k = 0, 1
    while k <= n:
        total, k = total + term(k), k + 1
    return total
```

Function of a single argument (*not called "term"*)

A formal parameter that will be bound to a function

The cube function is passed as an argument value

0 + 1 + 8 + 27 + 64 + 125

The function bound to term gets called here

Functions as Return Values

(Demo)

Locally Defined Functions

Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

```
def make_adder(n):
    """Return a function that takes one argument k and returns k + n.

    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
```

Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

A function that returns a function

```
def make_adder(n):
    """Return a function that takes one argument k and returns k + n.

    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
```

Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

A function that returns a function

```
def make_adder(n):
    """Return a function that takes one argument k and returns k + n.

    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
```

The name add_three is bound to a function

Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

A function that returns a function

```
def make_adder(n):
    """Return a function that takes one argument k and returns k + n.

    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
```

The name add_three is bound to a function

A def statement within another def statement

Locally Defined Functions

Functions defined within other function bodies are bound to names in a local frame

A function that returns a function

```
def make_adder(n):
    """Return a function that takes one argument k and returns k + n.

    >>> add_three = make_adder(3)
    >>> add_three(4)
    7
    """
    def adder(k):
        return k + n
    return adder
```

The name add_three is bound to a function

A def statement within another def statement

Can refer to names in the enclosing function

Call Expressions as Operator Expressions

Call Expressions as Operator Expressions

```
make_adder(1)    (        2        )
```

Call Expressions as Operator Expressions

Operator

```
make_adder(1)    (        2        )
```

Call Expressions as Operator Expressions

Operator		*Operand*	
`make_adder(1)`	`(`	`2`	`)`

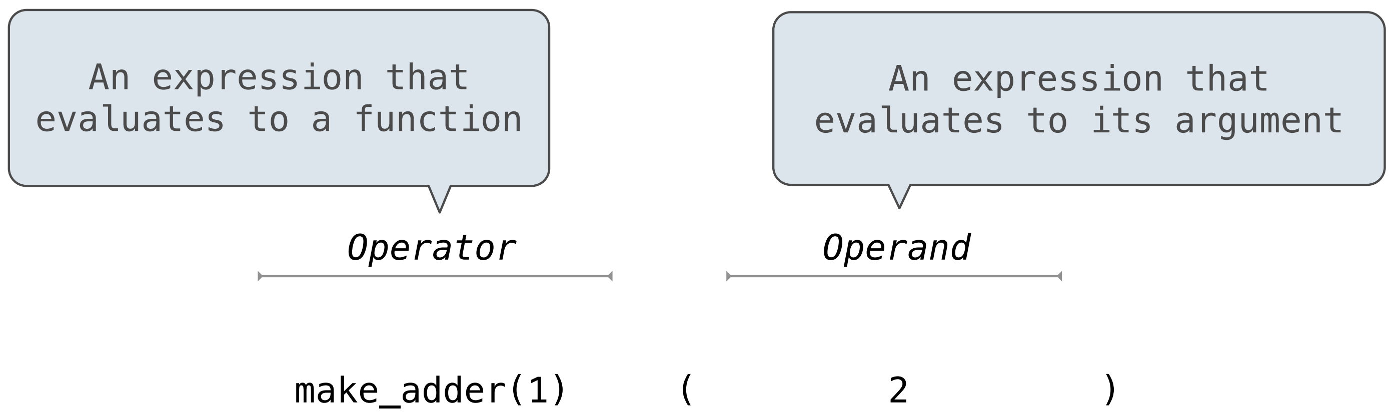
Call Expressions as Operator Expressions

An expression that
evaluates to a function

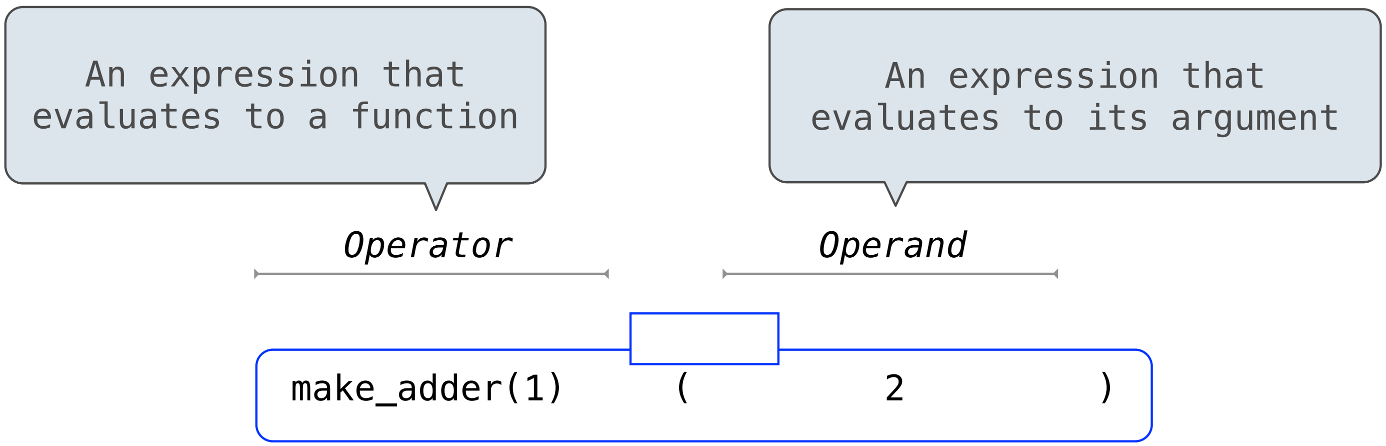
Operator *Operand*

```
make_adder(1)    (        2         )
```

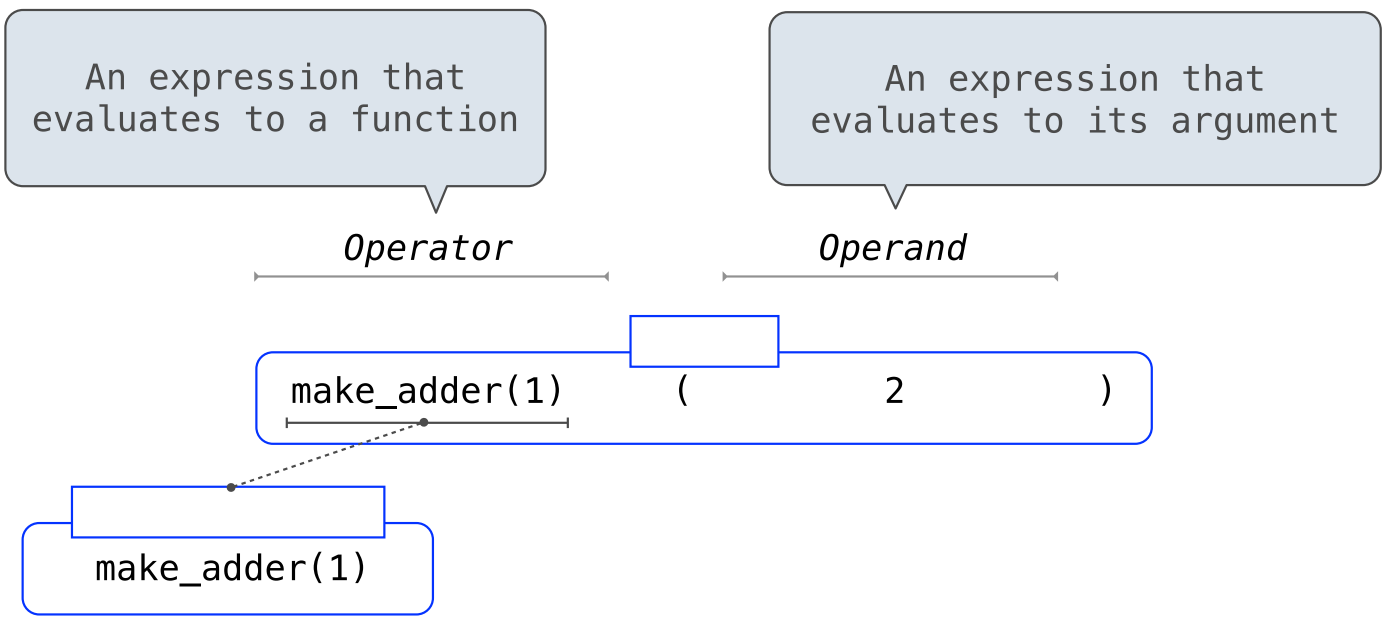
Call Expressions as Operator Expressions



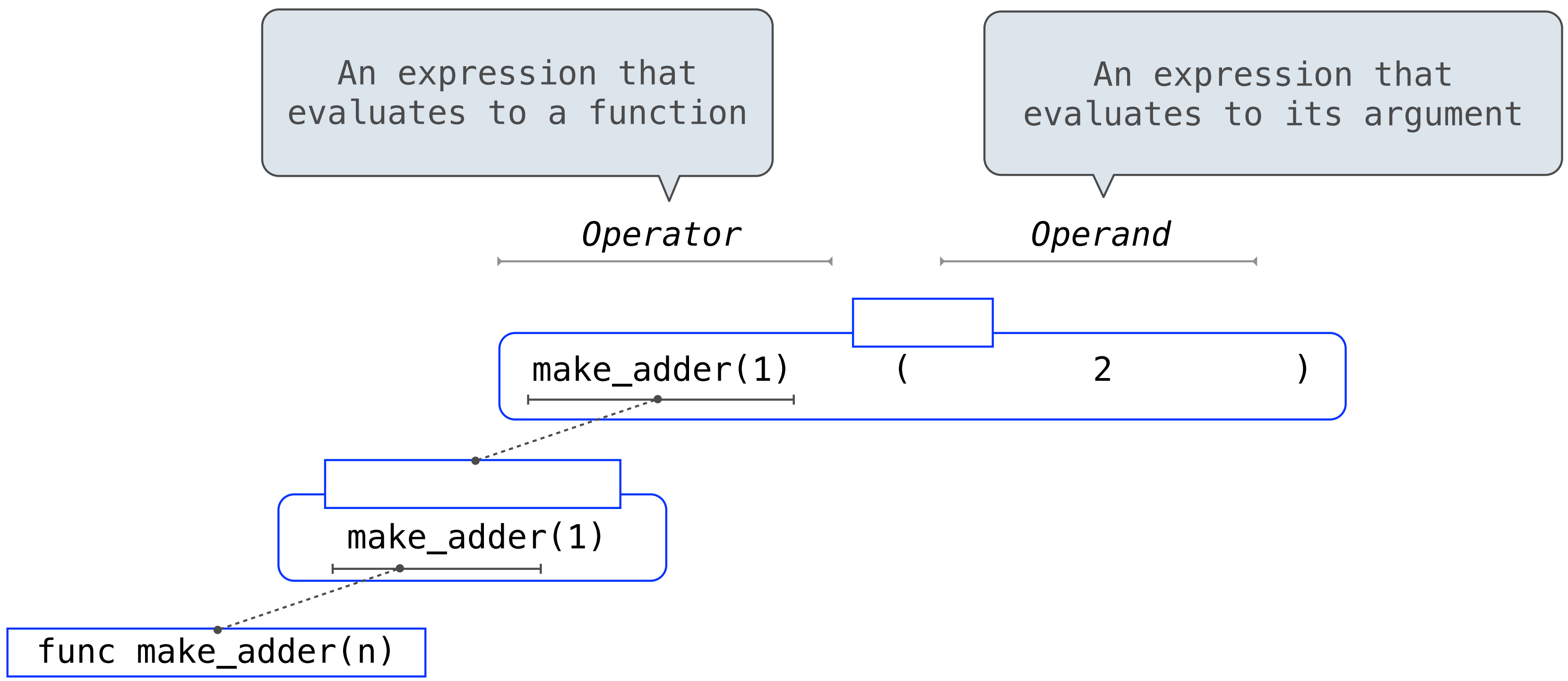
Call Expressions as Operator Expressions



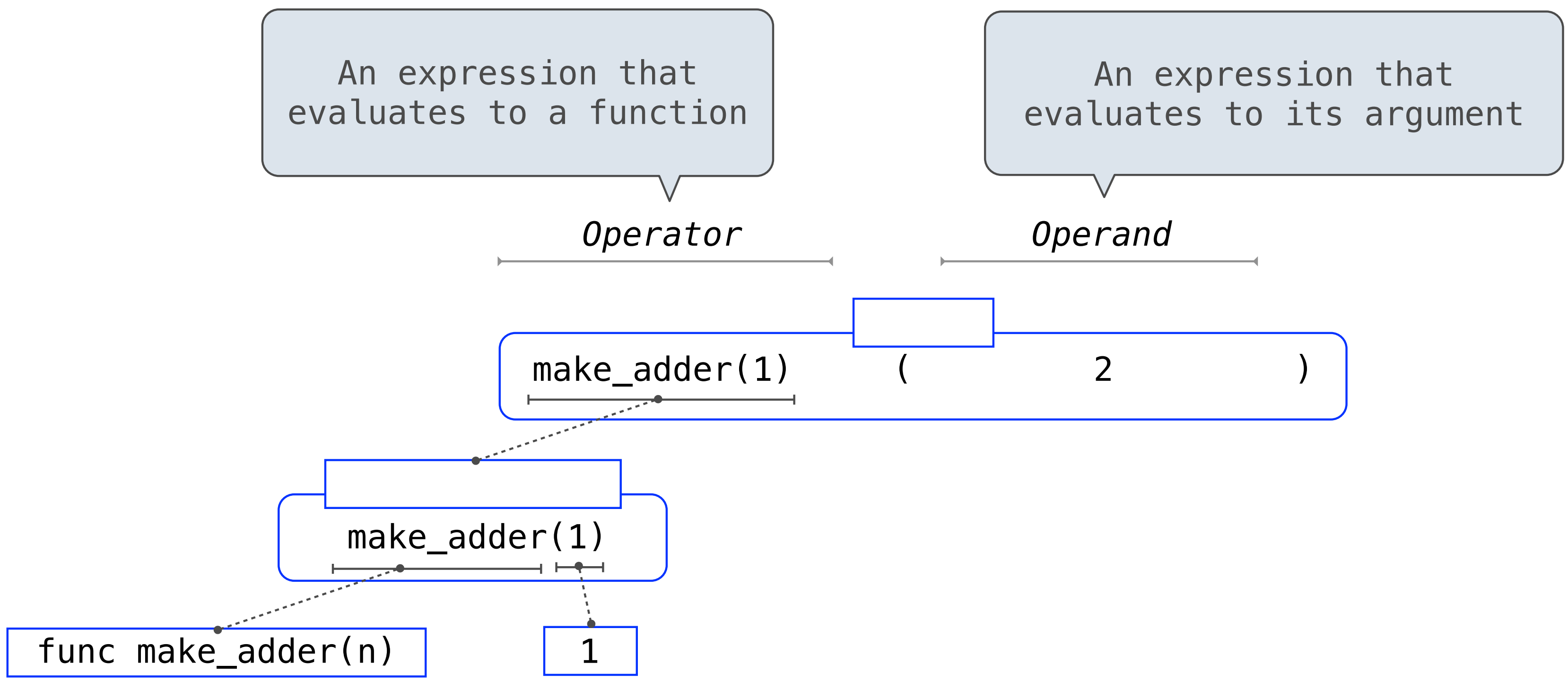
Call Expressions as Operator Expressions



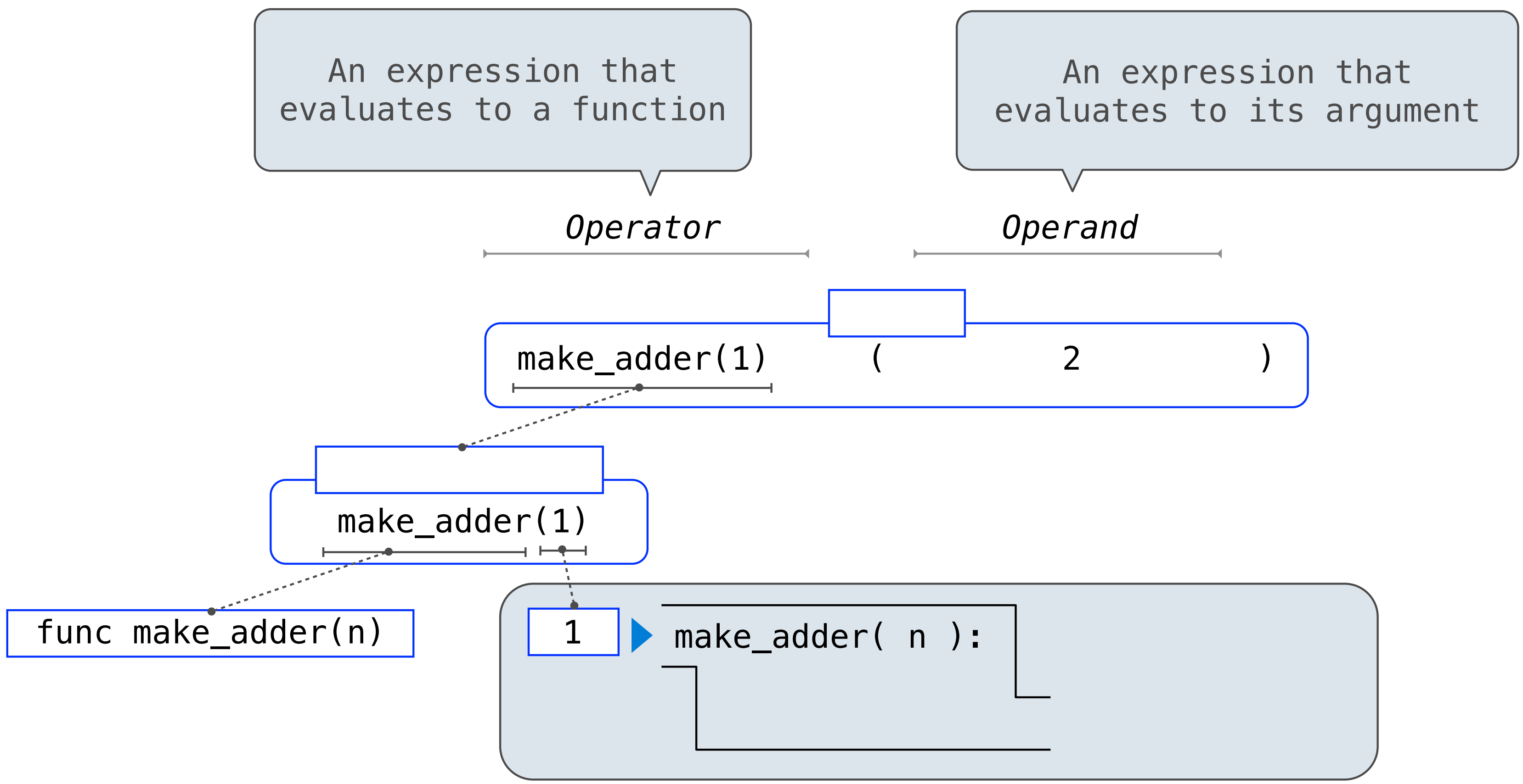
Call Expressions as Operator Expressions



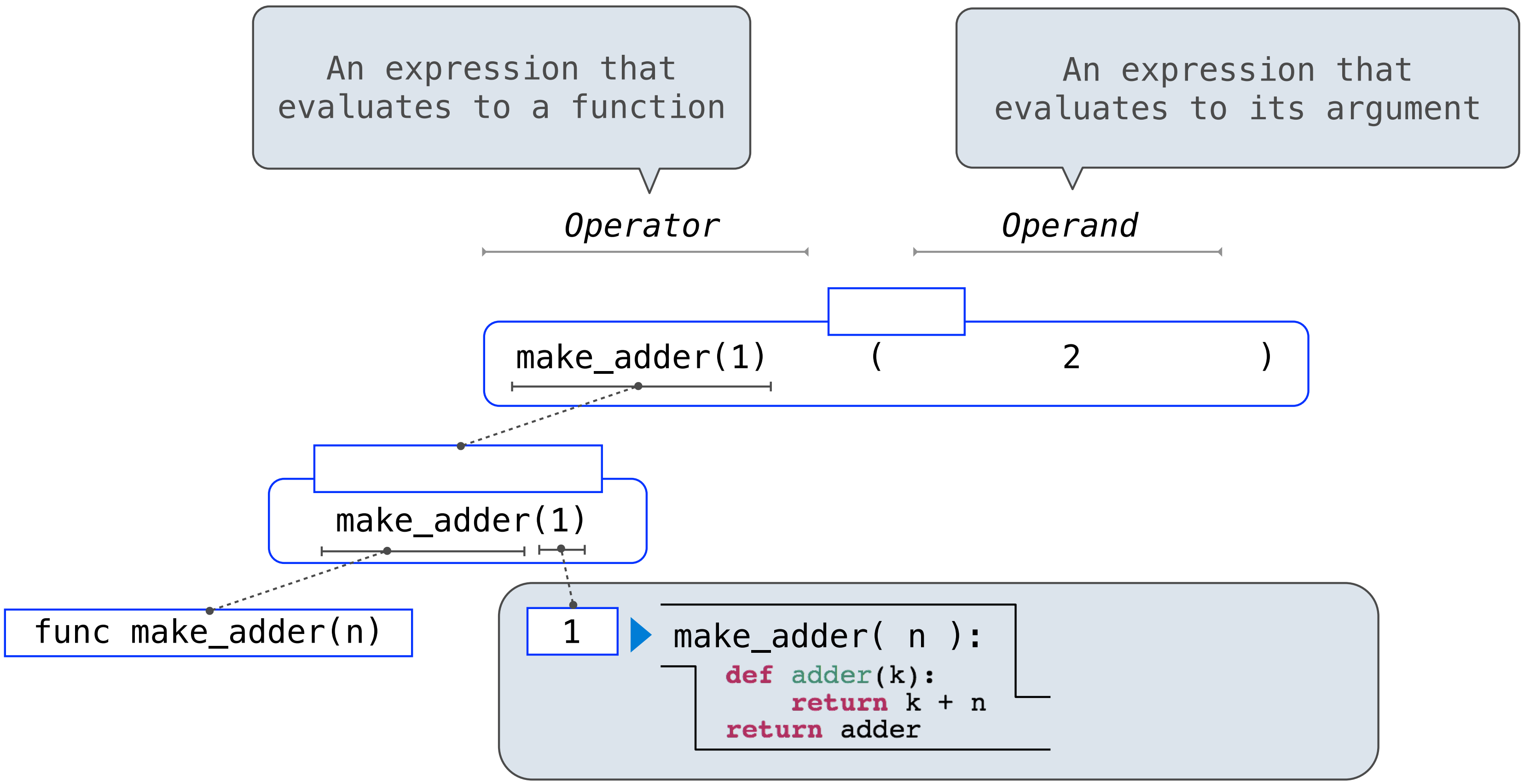
Call Expressions as Operator Expressions



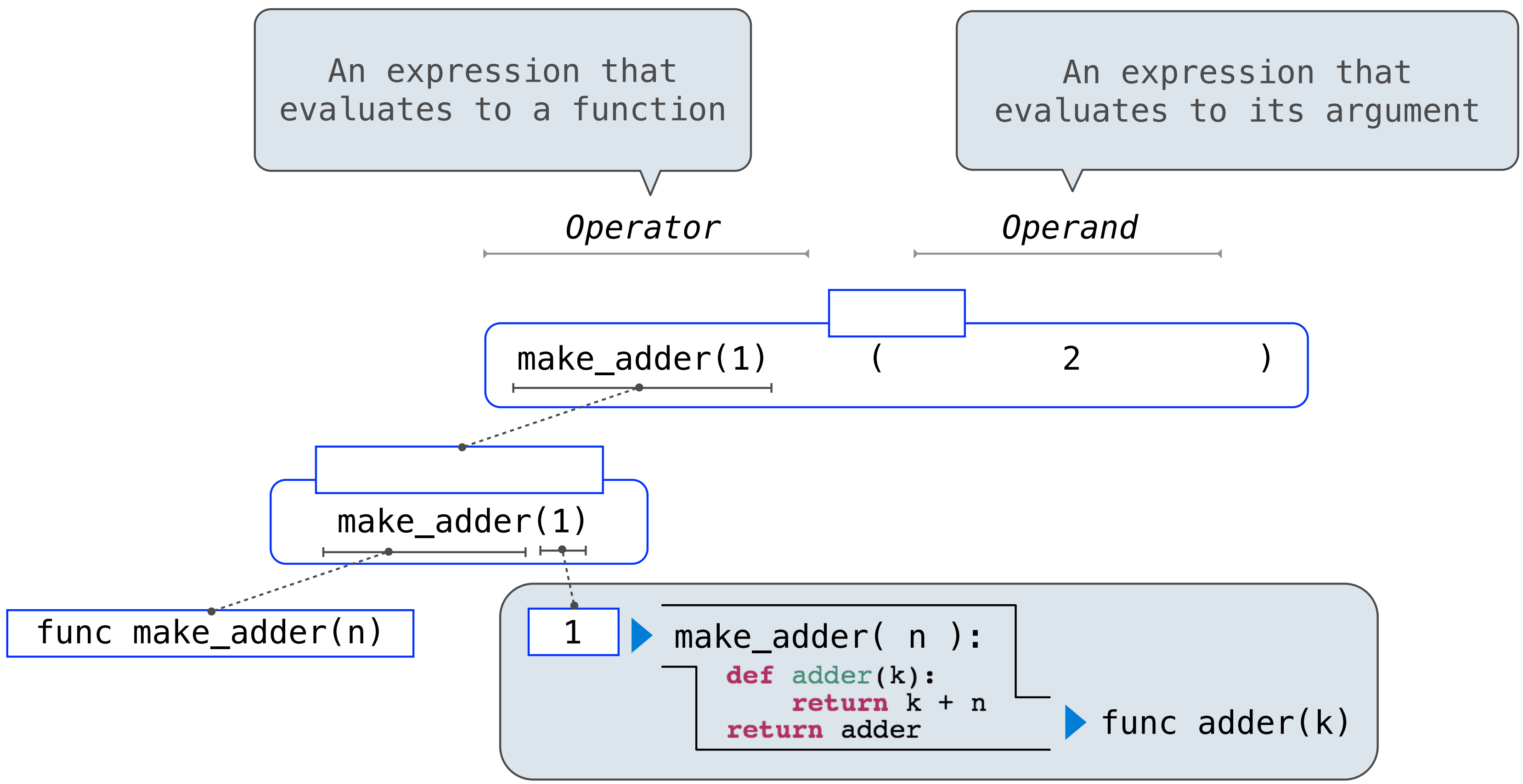
Call Expressions as Operator Expressions



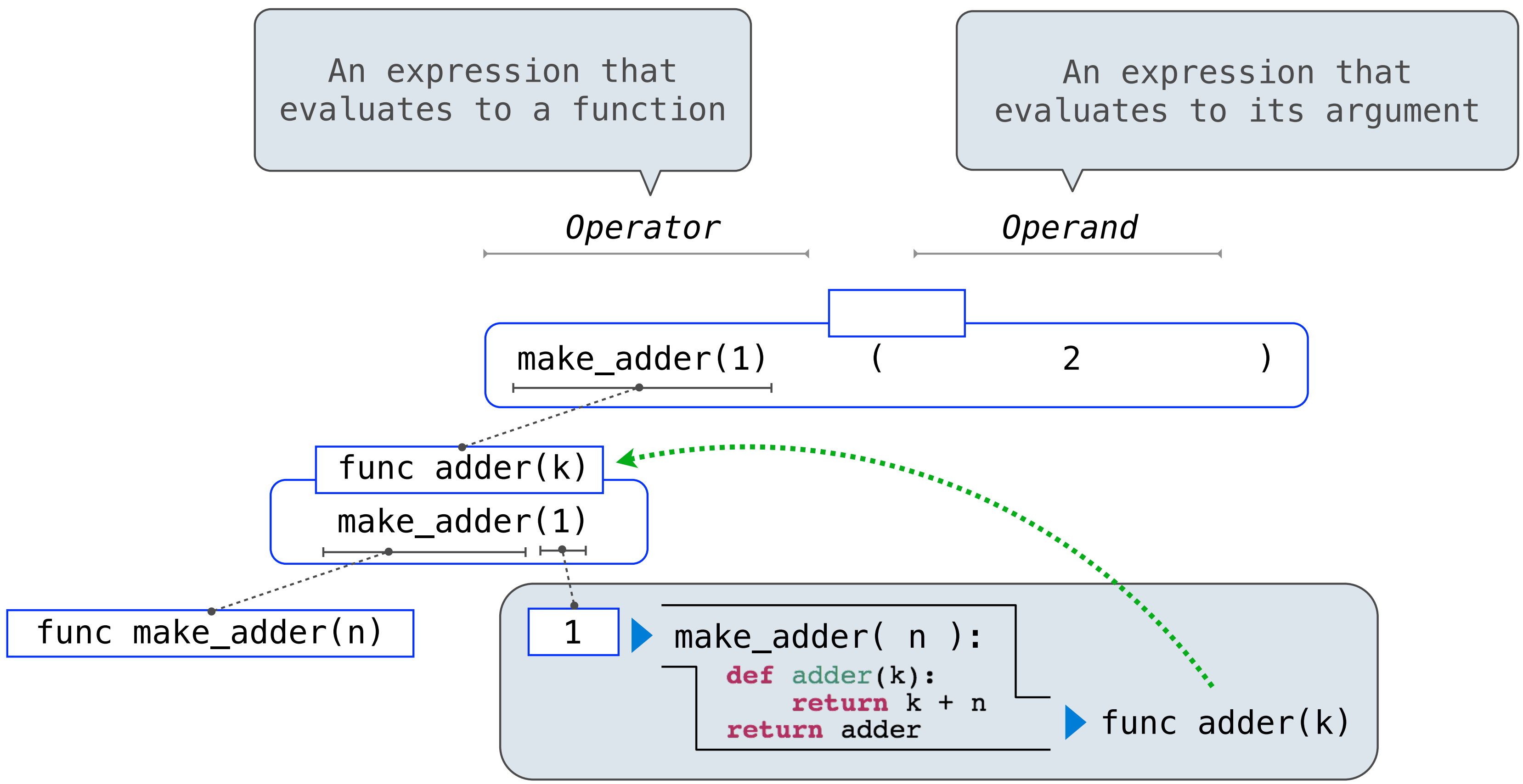
Call Expressions as Operator Expressions



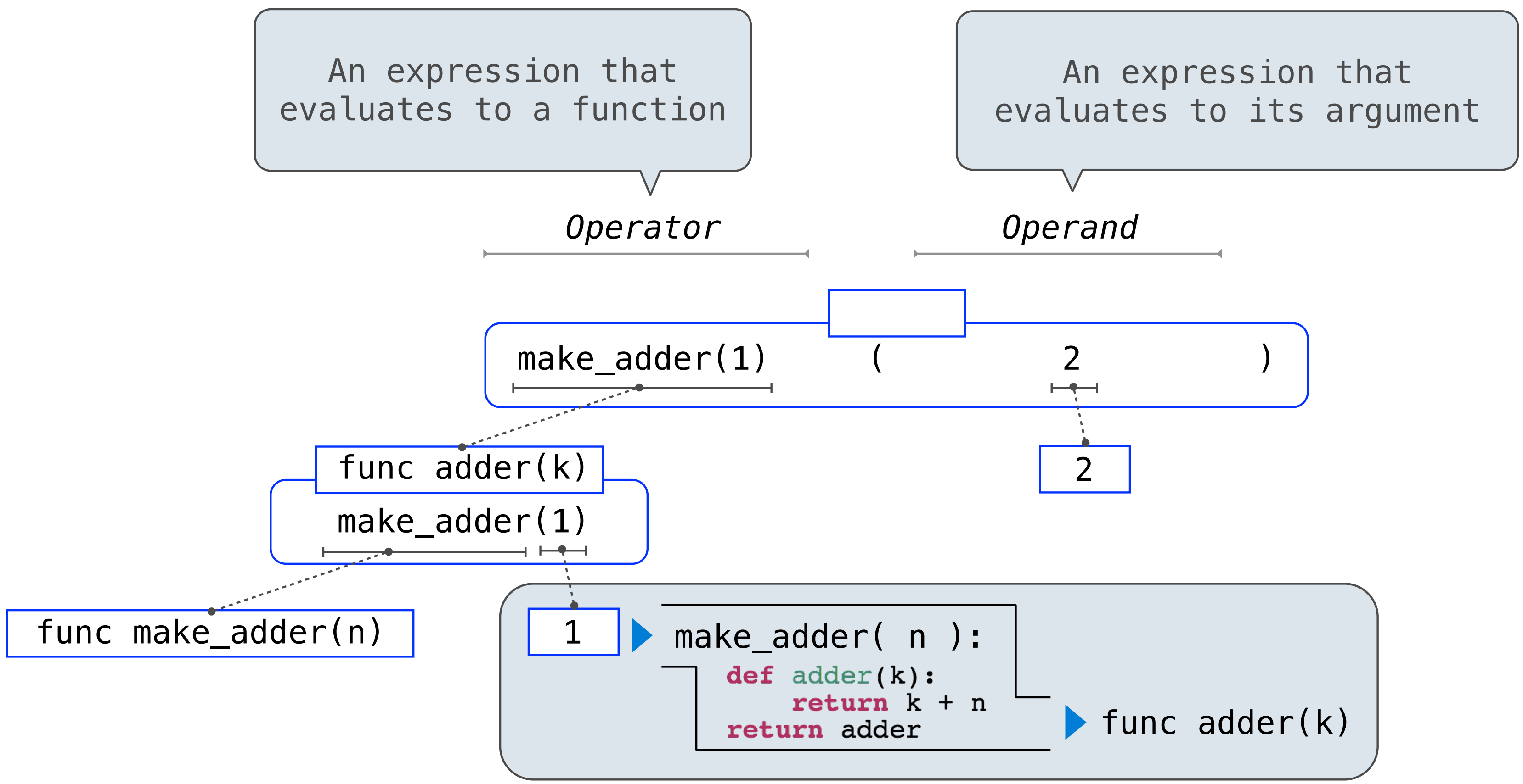
Call Expressions as Operator Expressions



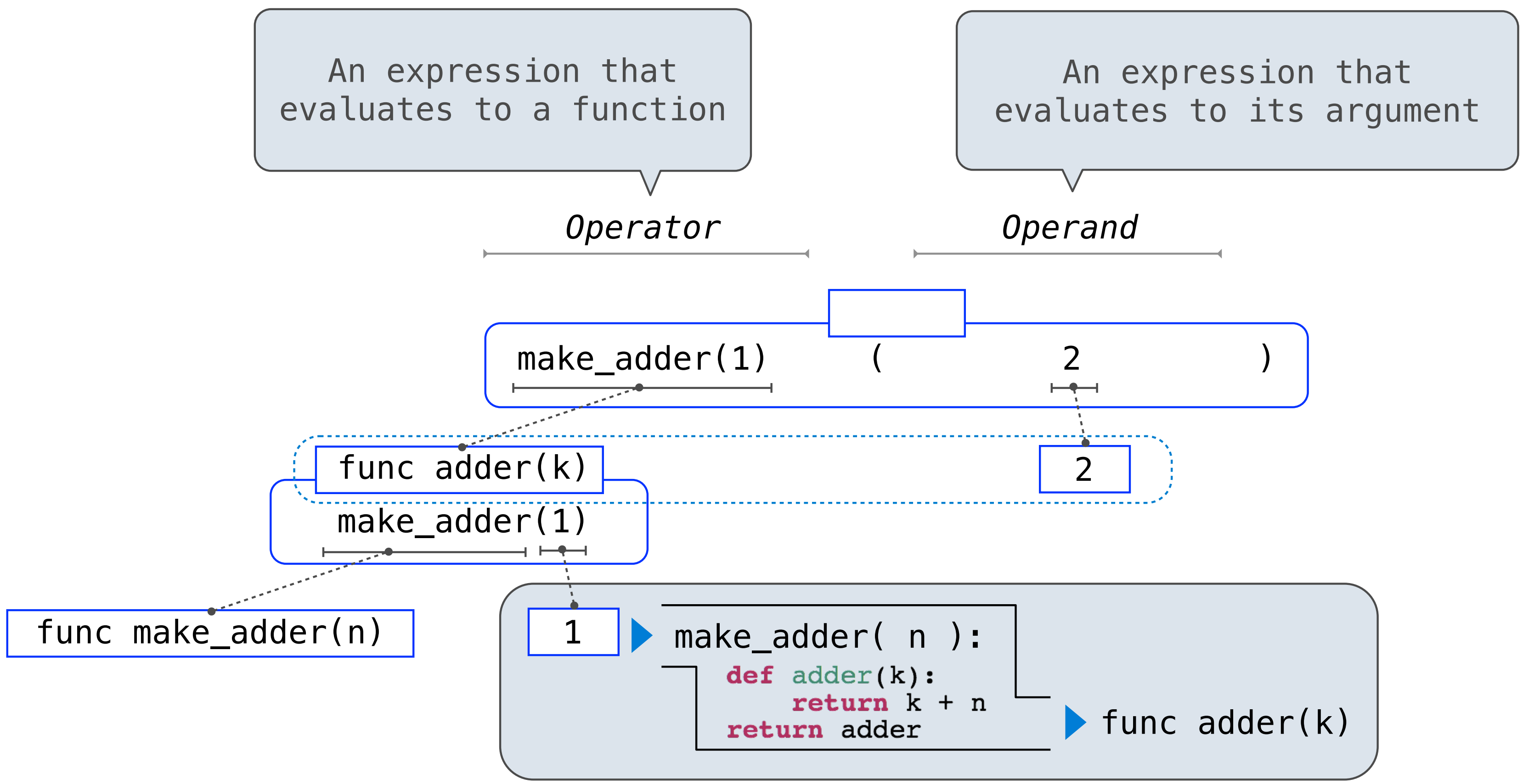
Call Expressions as Operator Expressions



Call Expressions as Operator Expressions



Call Expressions as Operator Expressions



Call Expressions as Operator Expressions