CS 189 Introduction to Machine Learning
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Exam Prep 1

This exam-prep discussion section covers Bayesian decision theory and maximum likelihood estimation. In order, the questions were taken from the Spring offerings in 2016, 2016, 2017, 2019, and 2017.

1 Multiple Choice

(f) [3 pts] The Bayes risk for a decision problem is zero when

○ the class distributions $P(X|Y)$ do not overlap.

○ the loss function $L(z, y)$ is symmetrical.

○ the Bayes decision rule perfectly classifies the training data.

○ the training data is linearly separable.

(g) [3 pts] Let $L(z, y)$ be a loss function (where y is the true class and z is the predicted class). Which of the following loss functions will *always* lead to the same Bayes decision rule as L?

○ $L_1(z, y) = aL(z, y)$, $a > 0$

○ $L_3(z, y) = L(z, y) + b$, $b > 0$

○ $L_2(z, y) = aL(z, y)$, $a < 0$

○ $L_4(z, y) = L(z, y) + b$, $b < 0$

(t) [3 pts] Which of the following statements about maximum likelihood estimation are true?

○ MLE, applied to estimate the mean parameter μ of a normal distribution $\mathcal{N}(\mu, \Sigma)$ with a known covariance matrix Σ, returns the mean of the sample points

○ MLE, applied to estimate the covariance parameter Σ of a normal distribution $\mathcal{N}(\mu, \Sigma)$, returns $\hat{\Sigma} = \frac{1}{n} X^T X$, where X is the design matrix

○ For a sample drawn from a normal distribution, the likelihood $\mathcal{L}(\mu, \sigma; X_1, \ldots, X_n)$ is equal to the probability of drawing exactly the points $X_1, \ldots, X_n$ (in that order) when you draw n random points from $\mathcal{N}(\mu, \sigma)$

○ Maximizing the log likelihood is equivalent to maximizing the likelihood

2 Free Response

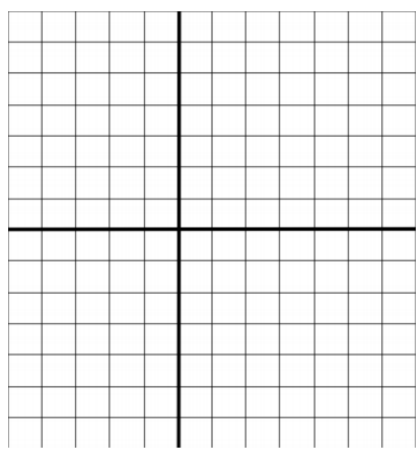
Q3. [10 pts] Quadratic Discriminant Analysis

(a) [4 pts] Consider 12 labeled data points sampled from three distinct classes:

Class 0: $\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \end{bmatrix}$ Class 1: $\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} 4\sqrt{2} \\ -\sqrt{2} \end{bmatrix}, \begin{bmatrix} -4\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$ Class 2: $\begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

For each class $C \in \{0, 1, 2\}$, compute the class sample mean μ_C, the class sample covariance matrix Σ_C, and the estimate of the prior probability π_C that a point belongs to class C. (Hint: $\mu_1 = \mu_0$ and $\Sigma_2 = \Sigma_0$.)

(b) [4 pts] Sketch one or more isocontours of the QDA-produced normal distribution or quadratic discriminant function (they each have the same contours) for each class. The isovalues are not important; the important aspects are the centers, axis directions, and relative axis lengths of the isocontours. Clearly label the centers of the isocontours and to which class they correspond.



(c) [2 pts] Suppose that we apply LDA to classify the data given in part (a). Why will this give a poor decision boundary?

Q3. [10 pts] Maximum Likelihood Estimation for Reliability Testing

Suppose we are reliability testing n units taken randomly from a population of identical appliances. We want to estimate the mean failure time of the population. We assume the failure times come from an exponential distribution with parameter $\lambda > 0$, whose probability density function is $f(x) = \lambda e^{-\lambda x}$ (on the domain $x \geq 0$) and whose cumulative distribution function is $F(x) = \int_0^x f(x)\,\mathrm{d}x = 1 - e^{-\lambda x}$.

(a) [6 pts] In an ideal (but impractical) scenario, we run the units until they all fail. The failure times are $t_1, t_2, \ldots, t_n$.

Formulate the likelihood function $\mathcal{L}(\lambda; t_1, \ldots, t_n)$ for our data. Then find the maximum likelihood estimate $\hat{\lambda}$ for the distribution's parameter.

(b) [4 pts] In a more realistic scenario, we run the units for a fixed time T. We observe r unit failures, where $0 \leq r \leq n$, and there are $n - r$ units that survive the entire time T without failing. The failure times are $t_1, t_2, \ldots, t_r$.

Formulate the likelihood function $\mathcal{L}(\lambda; n, r, t_1, \ldots, t_r)$ for our data. Then find the maximum likelihood estimate $\hat{\lambda}$ for the distribution's parameter.

Hint 1: What is the probability that a unit will not fail during time T? *Hint 2:* It is okay to define $\mathcal{L}(\lambda)$ in a way that includes contributions (densities and probability masses) that are not commensurate with each other. Then the constant of proportionality of $\mathcal{L}(\lambda)$ is meaningless, but that constant is irrelevant for finding the best-fit parameter $\hat{\lambda}$. *Hint 3:* If you're confused, for part marks write down the likelihood that r units fail and $n-r$ units survive; then try the full problem. *Hint 4:* If you do it right, $\hat{\lambda}$ will be the number of observed failures divided by the sum of unit test times.