CS 189 Introduction to Machine Learning
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Exam Prep 2

This exam-prep discussion section covers anisotropic normal distributions, Quadratic and Linear Discriminant Analyses, and some of their extensions.

1 Multiple Choice

(m) [3 pts] In LDA/QDA, what are the effects of modifying the sample covariance matrix as $\tilde{\Sigma} = (1 - \lambda)\Sigma + \lambda I$, where $0 < \lambda < 1$?

○ $\tilde{\Sigma}$ is positive definite

○ $\tilde{\Sigma}$ is invertible

○ Increases the eigenvalues of Σ by λ

○ The isocontours of the quadratic form of $\tilde{\Sigma}$ are closer to spherical

(h) [3 pts] We are using **linear discriminant analysis** to classify points $x \in \mathbb{R}^d$ into **three** different classes. Let S be the set of points in $\mathbb{R}^d$ that our trained model classifies as belonging to the first class. Which of the following are true?

○ The decision boundary of S is always a hyperplane

○ S can be the whole space $\mathbb{R}^d$

○ The decision boundary of S is always a subset of a union of hyperplanes

○ S is always connected (that is, every pair of points in S is connected by a path in S)

(o) [3 pts] Suppose you have a **multivariate normal distribution** with a positive definite covariance matrix Σ. Consider a second multivariate Gaussian distribution whose covariance matrix is $\kappa\Sigma$, where $\kappa = \cos\theta > 0$. Which of the following statements are true about the ellipsoidal isocontours of the second distribution, compared to the first distribution?

○ The principal axes of the ellipsoids would be rotated by θ

○ The principal axes (radii) of the ellipsoids will be scaled by $1/\kappa$

○ The principal axes (radii) of the ellipsoids will be scaled by κ

○ The principal axes (radii) of the ellipsoids will be scaled by $\sqrt{\kappa}$

(s) [3 pts] Suppose you have a sample in which each point has d features and comes from class C or class D. The class conditional distributions are $(X_i|y_i = C) \sim N(\mu_C, \sigma_C^2)$ and $(X_i|y_i = D) \sim N(\mu_D, \sigma_D^2)$ for unknown values $\mu_C, \mu_D \in \mathbb{R}^d$ and $\sigma_C^2, \sigma_D^2 \in \mathbb{R}$. The class priors are π_C and π_D. We use 0-1 loss.

○ If $\pi_C = \pi_D$ and $\sigma_C = \sigma_D$, then the Bayes decision rule assigns a test point z to the class whose mean is closest to z.

○ If $\sigma_C = \sigma_D$, then the Bayes decision boundary is always linear.

○ If $\pi_C = \pi_D$, then the Bayes decision rule is $r^*(z) = \text{argmin}_{A\in\{C,D\}} \left(|z - \mu_A|^2/(2\sigma_A^2) + d \ln \sigma_A\right)$

○ If $\sigma_C = \sigma_D$, then QDA will always produce a linear decision boundary when you fit it to your sample.

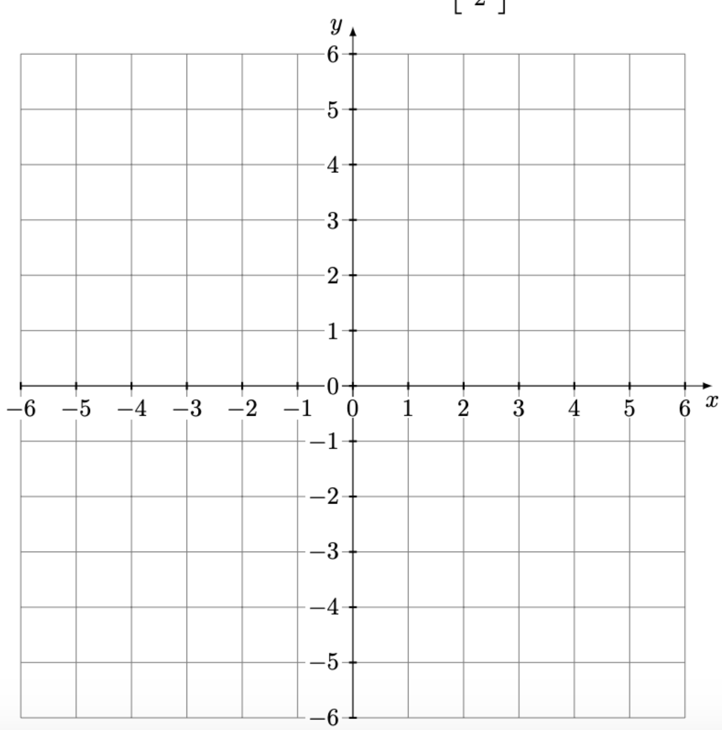
2 Quadratics and Gaussian Isocontours (Spring 2016)

(a) [4 pts] Write the 2×2 matrix Σ whose unit eigenvectors are $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ with eigenvalue 1 and $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ with eigenvalue 4. Write out **both** the eigendecomposition of Σ and the final 2×2 matrix Σ.

(b) [3 pts] Write the symmetric square root $\Sigma^{1/2}$ of Σ. (The eigendecomposition is optional, but it might earn you partial credit if you get $\Sigma^{1/2}$ wrong.)

(c) [3 pts] Consider the bivariate Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$. Let $P(X = \mathbf{x})$ be its probability distribution function (PDF). Write the formula for the isocontour $P(\mathbf{x}) = e^{-\sqrt{5}/2}/(4\pi)$, substitute in the value of the determinant $|\Sigma|$ from part (a) (but leave μ and Σ^{-1} as variables), and simplify the formula as much as you can.

(d) [5 pts] Draw the isocontour $P(\mathbf{x}) = e^{-\sqrt{5}/2}/(4\pi)$ where $\mu = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and Σ is given in part (a).



3 Discriminant Analysis (Spring 2016)

Let's derive the decision boundary when one class is Gaussian and the other class is exponential. Our feature space is one-dimensional ($d = 1$), so the decision boundary is a small set of points.

We have two classes, named N for normal and E for exponential. For the former class ($Y = N$), the prior probability is $\pi_N = P(Y = N) = \frac{\sqrt{2\pi}}{1+\sqrt{2\pi}}$ and the class conditional $P(X|Y = N)$ has the normal distribution $\mathcal{N}(0, \sigma^2)$. For the latter, the prior probability is $\pi_E = P(Y = E) = \frac{1}{1+\sqrt{2\pi}}$ and the class conditional has the exponential distribution

$$P(X = x|Y = E) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Write an equation in x for the decision boundary. (Only the positive solutions of your equation will be relevant; ignore all $x < 0$.) Use the 0-1 loss function. Simplify the equation until it is quadratic in x. (You don't need to solve the quadratic equation. It should contain the constants σ and λ. Ignore the fact that 0 might or might not also be a point in the decision boundary.) **Show your work**, starting from the posterior probabilities.